Pliant Arithmetics and Pliant Arithmetic Operations

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Abstract

Fuzzy arithmetic based α-cuts, where the result of the α-cuts represent an interval. The arithmetic can be understand as an interval arithmetic of the α-cuts. Instead of dealing with intervals we are dealing with left and right hand sided soft inequalities which define the interval. We offer a new calculation procedure of arithmetics, when these soft inequalities meet certain properties (i.e. strict monotonously increasing function represent the inequality). We show that the result of linear combinations of linear is also linear and the linear combination of sigmoid is also sigmoid function (i.e. they are closed under linear combination). We give the result of other operation, too. The soft inequalities define an interval by using proper conjunctive and disjunctive operator. We give such operations, too.

Keywords: Fuzzy arithmetic, sigmoid function, triangular membership function, membership function, distending function.

1 Introduction

The idea the fuzzy quantities could be arithmetically combined according to the laws of fuzzy set theory is due to Zadeh [14]. Soon after, several researchers worked independently along these lines, such as Jain [5], Mizumoto and Tanaka [9, 10], Nahmias [11], Nguyen [12], Dubois and Prade [1]. It was only further on recognized that the mathematics of fuzzy quantities are an application of possibility theory, an extension of interval analysis as well as of the algebra of many-values quantities (Young [13]).

Fuzzy interval extends and updates the overview of Dubois and Prade [2]. Several theoretical details and applications can be found e.g., in monographs of Kaufmann and Gupta [6, 7], and Mares [8]. In 1987, teher was a special issue of Fuzzy Sets and Systems (Dubois and Prade [3]) devoted to the fuzzy intervals domain, and more recently another one has appeared (Fullér and Mesiar [4]).

In real world applications we often need to deal with imprecise quantities. They can be results of measurements or vague statements, e.g. I have about 40 dollars in my pocket, she is approximately 170cm tall. In arithmetics we can use $a < x$ and $x < b$ inequalites to characterize such quantities, e.g. if I have about 40 dollars then my money is probably more than 35 dollars and less than 45 dollars.

Fuzzy numbers can also be used to represent imprecise quantities. Pliant numbers are created by *softening* the $a < x$ and $x < b$ inequalities, i.e. replacing the crisp characteristic function with two fuzzy membership functions and applying a fuzzy conjunction operator to combine the two functions. We refer to the softened inequalities as *fuzzy inequalities*.

We call the distending function corresponding to the $x < a$ interval the left side of the fuzzy number and denote it as δ_l. Similarly we refer to the distending function corresponding to the $x < b$ interval as the right side of the fuzzy number and denote it as δ_r.

We will use the following terminology: function representing the soft inequality called distending function. The word pliant means flexible instead of using distending we use soft inequalities and additive pliant is when $f_c(x) + f_d(x) = 1$ (at nilpotent operator case) and multiplicative pliant if $f_c(x)f_d(x) = 1$ (at strict monotone operator case).

Naturally one would like to execute arithmetic operations over fuzzy numbers. Fuzzy arithmetic operations are generally carried out using the α-cut method. In Section 2 we propose a new and efficient method for arithmetic calculations. The next two sections discuss the arithmetic operations and their properties for two classes of fuzzy distending functions. Section 3 investigates additive pliant functions, i.e. distending functions represented as lines. Section 4 presents multiplicative pliant functions, i.e. distending functions based on pliant inequalities. Finally, Section 5 examines which conjunction operators are suitable for constructing fuzzy numbers from additive and multiplicative pliant.

2 Fuzzy arithmetics

Fuzzy arithmetic operations are based on the extension principle of arithmetics. In arithmetics we can find the result of an arithmetic operation by measuring the distance of the operands from the zero point than applying the operation on these distances. Fig. 1 presents this idea in case of addition.

In fuzzy arithmetics we deal with fuzzy numbers. Fuzzy numbers are mappings from real numbers to the $[0, 1]$ real interval. Operations are executed by creating an α-cut for all $\alpha \in [0, 1]$ and using the arithmetic principle to get the resulting value for each α value. Fig. 2 demonstrates fuzzy addition with fuzzy numbers represented as lines. The dotted triangle number

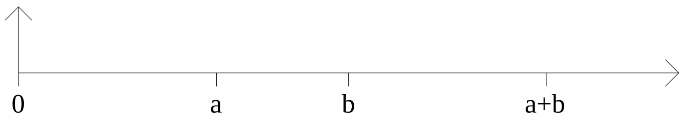

Figure 1: Arithmetic addition

is the sum of the two other triangle numbers.

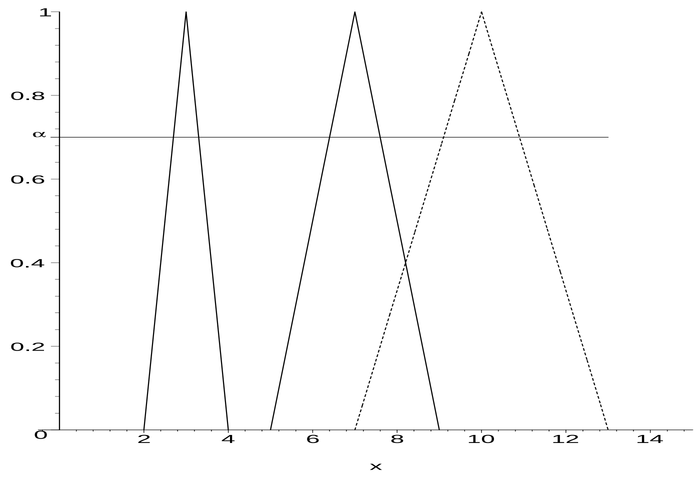

Figure 2: Fuzzy addition with α-cut.

This way we can have all the well-known unary ($-x$, x^2) and binary operations ($x + y$, xy, $x \, mod \, y$) available as fuzzy operations. However the calculation of fuzzy operations with α-cut is tedious and often impractical. In this paper a new efficient method is proposed which is equivalent with the α-cut.

Fuzzy numbers are often composed of two strictly monotone functions, i.e. the left side denoted as δ_l, and the right side denoted as δ_r of the fuzzy number. Fuzzy operations can be carried out by first applying them to the left sides than to the right sides of the operands.

This separation allows us to treat fuzzy numbers as strictly monotone functions when dealing with fuzzy arithmetic operations. In the following we omit the subscript from δ_l and δ_r and simply write δ with the inherent assumption that we shall only do arithmetic operations with functions representing the same side of fuzzy numbers.

Lemma 2.1. *Let $\delta_1, \delta_2, \ldots, \delta_n$ ($n \geq 1$) be strictly monotone functions representing soft inequalities and let F be an n-ary fuzzy operation over them. If*

$$\delta = F(\delta_1, \delta_2, \ldots, \delta_n),$$

then

$$\begin{aligned} \delta(z) &= \left(F\left({\delta_1}^{-1}, {\delta_2}^{-1}, \ldots, {\delta_n}^{-1}\right)\right)^{-1}(z) \iff \qquad (1) \\ \delta(z) &= \sup_{F(x_1,x_2,\ldots,x_n)=z} \min\left\{\delta_1(x_1), \delta_2(x_2), \ldots, \delta(x_n)\right\}. \end{aligned}$$

Proof. It can be easily verified that the method is equivalent with the α-cut. □

Fig. 3 visualizes the equivalence for the addition of lines. The left side shows the result of lines added together using α-cut with the result presented as the dotted line. On the right side we have simply added together the inverse functions of the two operands. The result is also presented as a dotted line. It can be seen from the figures that the result of the α-cut is indeed the inverse of the result in the right hand side figure.

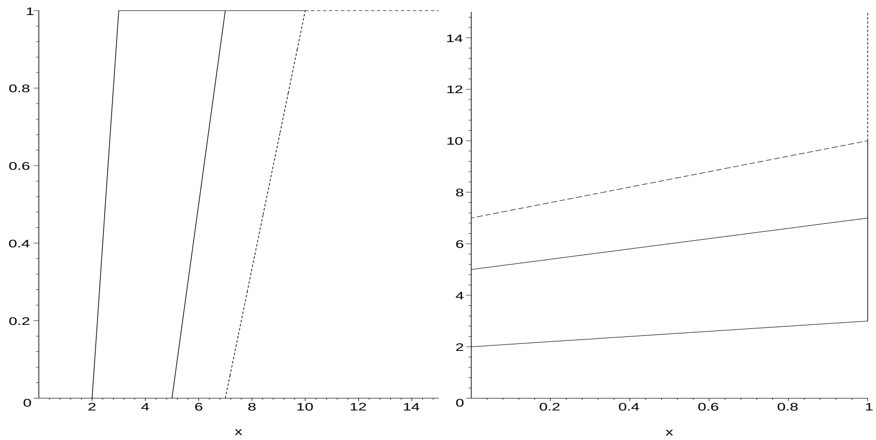

Figure 3: Left: α-cut addition, right: inverse of addition.

We can state a theorem regarding the properties of fuzzy operations.

Theorem 2.2. *Let $\delta_1, \delta_2, \ldots, \delta_n$ ($n \geq 1$) be strictly monotone functions representing fuzzy inequalities and let F be an n-ary fuzzy operation over them. If*

$$F\left({\delta_1}^{-1}, {\delta_2}^{-1}, \ldots, {\delta_n}^{-1}\right)$$

is strictly monotone then F has all the properties as its non-fuzzy interpretation.

Proof. It can be derived from Eq. 1 in Lemma 2.1. □

3 Additive Pliant

Triangle fuzzy numbers are commonly used to represent approximate values. A triangle fuzzy number has one line on each side. We can add triangle fuzzy numbers by first adding their left lines and than adding their right lines together.

Lemma 2.1 let us derive a general formula for adding lines.

Definition 3.1. *We say that a line $l(x)$ is given by its mean value if*

$$l(x) = m(x-a)+\frac{1}{2}$$

as shown in Fig. 4.

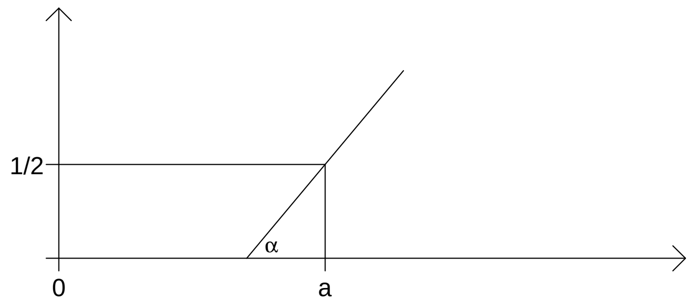

Figure 4: Line given by its mean value a and tangent $m = \tan\alpha$.

The inverse of $l(x)$ denoted as $l^{-1}(y)$ can be calculated easily

$$l^{-1}(y) = \frac{y-\frac{1}{2}}{m}+a.$$

3.1 Addition

Theorem 3.2. *Let $l_i(x) = m_i(x-a_i)+\frac{1}{2}$ ($i \in \{1,\dots,n\}$) lines given by their mean values. The fuzzy sum of l_i lines denoted as l is also a line and can be given as*

$$l(x) = l_1(x)\oplus\ldots\oplus l_n(x) = m(x-a)+\frac{1}{2}$$

where

$$\frac{1}{m} = \sum_{i=1}^{n}\frac{1}{m_i} \quad \textit{and} \quad a = \sum_{i=1}^{n} a_i.$$

Proof. Using Lemma 2.1 gives us

$$\begin{aligned} l^{-1}(y) &= \left(l_1{}^{-1}(y) + \ldots + l_n{}^{-1}(y)\right) \\ &= \sum_{i=1}^{n} \left(\frac{y - \frac{1}{2}}{m_i} + a_i \right) = \sum_{i=1}^{n} \left(\frac{y - \frac{1}{2}}{m_i} \right) + \sum_{i=1}^{n} a_i = \\ &= \left(y - \frac{1}{2} \right) \sum_{i=1}^{n} \frac{1}{m_i} + \sum_{i=1}^{n} a_i \end{aligned}$$

From here we have

$$l(x) = \frac{1}{\sum_{i=1}^{n} \frac{1}{m_i}} \left(x - \sum_{i=1}^{n} a_i \right) + \frac{1}{2}.$$

Substituting $\frac{1}{m}$ and a into the equation we get the desired result

$$l(x) = m\,(x - a) + \frac{1}{2}.$$

□

3.2 Multiplication by scalar

Theorem 3.3. *Let*

$$l(x) = m(x - a) + \frac{1}{2}$$

line given by their mean values.

The scalar multiplication of the lines is:

$$c \odot l(x) = m'(x - a') + \frac{1}{2}$$

where

$$a' = ca \qquad m' = \frac{m}{c}$$

Proof. Using Lemma 2.1 gives us

$$l^{-1}(x) = c \left(\frac{y - \frac{1}{2}}{m} + a \right).$$

From here we have

$$l(x) = \frac{m}{c}(x - ca) + \frac{1}{2}.$$

□

3.3 Subtraction

Calculations for subtraction yields

$$l(x) = l_1 \ominus l_2 = \frac{1}{\frac{1}{m_1} - \frac{1}{m_2}} (x - (a_1 - a_2)) + \frac{1}{2}.$$

Note: l does not exist when $m_1 = m_2$.
It is an important property that the result of the operation is also a line, i.e. the operation is closed for lines.

3.4 Multiplication and the n$^{\text{th}}$ power

Now let us calculate $\delta = l_1 \otimes l_2$, the product of l_1 and l_2.

$$\begin{aligned}
\delta(y)^{-1} &= \left(l_1{}^{-1} l_2{}^{-1}\right) = \left(\frac{y - \frac{1}{2}}{m_1} + a_1\right)\left(\frac{y - \frac{1}{2}}{m_2} + a_2\right) \\
&= \frac{(y - \frac{1}{2})^2}{m_1 m_2} + \left(\frac{a_1}{m_2} + \frac{a_2}{m_1}\right)\left(y - \frac{1}{2}\right) + a_1 a_2 \\
&= \left(\frac{(y - \frac{1}{2})}{\sqrt{m_1 m_2}} + \frac{(\frac{a_1}{m_2} + \frac{a_2}{m_1})\sqrt{m_1 m_2}}{2}\right)^2 - \frac{\left(\frac{a_1}{m_2} + \frac{a_2}{m_1}\right)^2 m_1 m_2}{4} + a_1 a_2 \\
&= \left(\frac{(y - \frac{1}{2})}{\sqrt{m_1 m_2}} + \frac{\frac{1}{2}(m_1 a_1 + m_2 a_2)}{\sqrt{m_1 m_2}}\right)^2 - \frac{(m_1 a_1 + m_2 a_2)^2}{4 m_1 m_2} + \frac{4 m_1 a_1 m_2 a_2}{4 m_1 m_2} \\
&= \frac{\left(y + \frac{1}{2}(m_1 a_2 + m_2 a_2) - \frac{1}{2}\right)^2}{m_1 m_2} - \frac{\frac{1}{4}(m_1 a_1 - m_2 a_2)^2}{m_1 m_2}.
\end{aligned}$$

From here we have

$$\delta(x) = \sqrt{m_1 m_2 x + \frac{1}{4}(m_1 a_1 - m_2 a_2)^2} - \frac{1}{2}(m_1 a_1 + m_2 a_2) + \frac{1}{2}.$$

Fig. 5 shows the result of multiplying two lines. The parameters were $m_1 = \frac{1}{5}, a_1 = 4$ and $m_2 = \frac{1}{3}, a_2 = 6$. The multiplication function is shown as the dotted curve.
The result is not a line. We need to remain in the world of lines to be able to carry out further arithmetic operations. This can be achieved by approximating the result with a line, e.g. using the least squares method.
Let us calculate the n$^{\text{th}}$ power of $l(x) = m(x - a) + \frac{1}{2}$.

$$\delta(y)^{-1} = \left(l^{-1}(y)\right)^n = \left(\frac{y - \frac{1}{2}}{m} + a\right)^n.$$

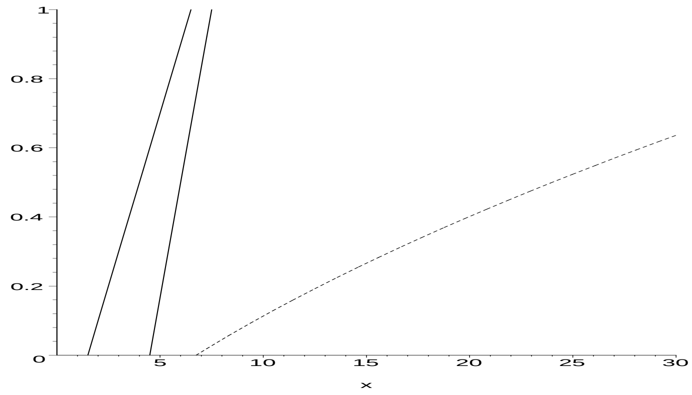

Figure 5: Multiplication of lines.

From here we get

$$\delta(x) = m\left(\sqrt[n]{x} - a\right) + \frac{1}{2}.$$

An approximation method should also be used here to get a line function from the result.

3.5 Properties of Operations

Theorem 3.4. *Addition is commutative and associative over lines.*

Proof. The properties can be easily seen from the construction of $\frac{1}{m}$ and a in Theorem 3.2. □

Theorem 3.5. *Multiplication over lines is commutative, associative and distributive over addition.*

Proof. Theorem 2.2 guarantees that these properties holds. □

4 Multiplicative Pliant

Let us start by introducing a special fuzzy inequality, the *pliant inequality* and examine its most important properties.

4.1 Pliant Inequality Model

Definition 4.1. *A pliant inequality is given as a sigmoid function of*

$$\{a <_\lambda x\} = \frac{1}{1 + e^{-\lambda(x-a)}} = \sigma_a^{(\lambda)}(x)$$

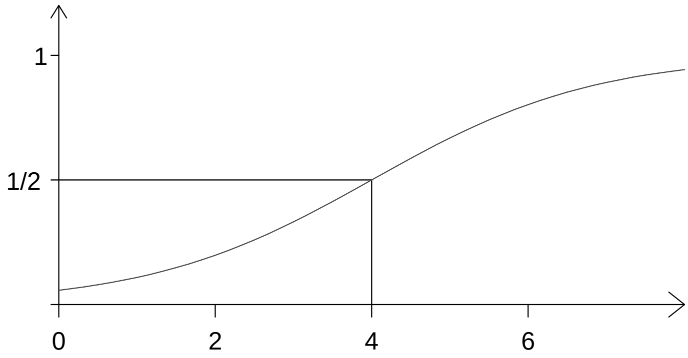

Figure 6: Pliant inequality with $\lambda = 0.7$ and $a = 4$ parameters.

where a is the mean value, i.e. $\sigma_a^{(\lambda)}(a) = \frac{1}{2}$.
The following properties can be seen from the figure

$$\begin{aligned} a < x \quad then \quad \{a <_\lambda x\} &> \frac{1}{2}, \\ a = x \quad then \quad \{a <_\lambda a\} &= \frac{1}{2}, \\ a > x \quad then \quad \{a <_\lambda x\} &< \frac{1}{2}. \end{aligned}$$

Definition 4.2. *The inverse function of* $\sigma_a^{(\lambda)}(x)$ *is denoted as* $\left(\sigma_a^{(\lambda)}\right)^{-1}(x)$ *and can be calculated easily. Let*

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}} = \omega,$$

then

$$\begin{aligned} 1 &= \omega(1 + e^{-\lambda(x-a)}) = \omega + \omega e^{-\lambda(x-a)} \\ \frac{1-\omega}{\omega} &= e^{-\lambda(x-a)} \\ \ln\left(\frac{1-\omega}{\omega}\right) &= -\lambda(x-a) \\ x = \left(\sigma_a^{(\lambda)}\right)^{-1}(\omega) &= \frac{1}{-\lambda}\ln\left(\frac{1-\omega}{\omega}\right) + a. \end{aligned}$$

Definition 4.3. *The first derivative of* $\sigma_a^{(\lambda)}(x)$ *is denoted as* $\left(\sigma_a^{(\lambda)}(x)\right)'$. *The following properties hold*

$$\left(\sigma_a^{(\lambda)}(a)\right)' = \frac{d\sigma_a^{(\lambda)}(x)}{dx}\bigg|_{x=a} = 4\lambda;$$

depending on λ, if

$$\begin{aligned}
\lambda > 0 \quad & \textit{then} \quad \left(\sigma_a^{(\lambda)}(x)\right)' \ \textit{is strictly monotone increasing,} \\
\lambda = 0 \quad & \textit{then} \quad \left(\sigma_a^{(\lambda)}(x)\right)' \equiv 0, \\
\lambda < 0 \quad & \textit{then} \quad \left(\sigma_a^{(\lambda)}(x)\right)' \ \textit{is strictly monotone decreasing.}
\end{aligned}$$

When we apply an arithmetic operation to pliant inequalities we need to make sure that the operation is meaningful, i.e. the pliant inequalities represent the same sides of the fuzzy numbers. The following criteria formulates this requirement.

Criteria 4.4. *If $\sigma_{a_1}^{(\lambda_1)}, \sigma_{a_2}^{(\lambda_2)}, \ldots, \sigma_{a_n}^{(\lambda_n)}$ are inputs to an n-ary fuzzy arithmetic operation then*

$$sgn(\lambda_1) = sgn(\lambda_2) = \ldots = sgn(\lambda_n)$$

must always hold.

4.2 Addition

Theorem 4.5. *Addition is closed over pliant inequalities and the addition function can be given as*

$$\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_n}^{(\lambda_n)} \quad = \quad \sigma_a^{(\lambda)} \qquad n \geq 1$$

where

$$\frac{1}{\lambda} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \quad \textit{and} \quad a = \sum_{i=1}^{n} a_i.$$

Proof. We prove by induction, if $i = 1$ then the statement is trivially true. Now let us assume that it holds for $i = n - 1$ and prove it for $i = n$,

$$\delta = \underbrace{\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_{n-1}}^{(\lambda_{n-1})}}_{\sigma_{a'}^{(\lambda')}} \oplus \sigma_{a_n}^{(\lambda_n)} = \sigma_{a'}^{(\lambda')} \oplus \sigma_{a_n}^{(\lambda_n)}$$

where

$$\frac{1}{\lambda'} = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \quad \text{and} \quad a' = \sum_{i=1}^{n-1} a_i.$$

Now by using Lemma 2.1 we have

$$\begin{aligned}\delta^{-1}(z) &= \left(\sigma_{a'}^{(\lambda')}\right)^{-1}(z) + \left(\sigma_{a_n}^{(\lambda_n)}\right)^{-1}(z) = \\ &= \frac{1}{-\lambda'}\ln\left(\frac{1-z}{z}\right) + a' + \frac{1}{-\lambda_n}\ln\left(\frac{1-z}{z}\right) + a_n = \\ &= \left(\sum_{i=1}^{n-1}\frac{1}{-\lambda_i} + \frac{1}{-\lambda_n}\right)\ln\left(\frac{1-z}{z}\right) + (\sum_{i=1}^{n-1} a_i + a_n) = \\ &= \frac{1}{-\lambda}\ln\left(\frac{1-z}{z}\right) + a. \qquad (2)\end{aligned}$$

If $\sum_{i=1}^{n}\frac{1}{\lambda_i} \neq 0$ then $\delta^{-1}(z)$ is a strictly monotone function and inverse of a pliant inequality. Therefore $\delta(x)$ is a pliant inequality with λ and a parameters:

$$\delta(x) = \left(\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_n}^{(\lambda_n)}\right)(x) = \frac{1}{1+e^{-\lambda}(x-(a))} = \sigma_a^{(\lambda)}(x). \qquad (3)$$

If $\sum_{i=1}^{n}\frac{1}{\lambda_i} = 0$ then the addition function does not exist since $\delta^{-1}(z) = a$ is a constant thus has no inverse. □

4.3 Multiplication by scalar

Theorem 4.6. *Let given $\sigma_a^{(\lambda)}(x)$ sigmoid function.*

The scalar multiplication of the sigmoid function is:

$$c \odot \sigma_a^{(\lambda)}(x) = \sigma_{a'}^{(\lambda')}(x)$$

where

$$\lambda' = \frac{\lambda}{c} \qquad a' = ca$$

Proof. Using Lemma 2.1 gives us

$$\left(\sigma_a^{(\lambda)}(x)\right)^{-1} = c\left(-\frac{1}{\lambda}\ln\left(\frac{1-x}{x}\right) + a\right)$$

From here we have

$$\sigma_{a'}^{(\lambda')}(x) = \frac{1}{1+e^{-\frac{\lambda}{c}(x-ca)}} = \frac{1}{1+e^{-\lambda'(x-a')}}$$

□

4.4 Subtraction

We can derive subtraction from addition and negation.

Lemma 4.7. *Negation is closed over pliant inequalities and the negation function can be given as*

$$\ominus\sigma_a^{(\lambda)} = \sigma_{-a}^{(-\lambda)}$$

Proof. Let

$$\delta = \ominus\sigma_a^{(\lambda)}$$

by using Lemma 2.1 we have

$$\begin{aligned}
\delta^{-1}(z) &= -\left(\left(\sigma_a^{(\lambda)}\right)^{-1}(z)\right) = \\
&= \frac{1}{\lambda}\ln\left(\frac{1-z}{z}\right) - a = -\frac{1}{-\lambda}\ln\left(\frac{1-z}{z}\right) + (-a).
\end{aligned}$$

Therefore

$$\delta(x) = \ominus\sigma_a^{(\lambda)}(x) = \frac{1}{1+e^{-(-\lambda)(x-(-a))}} = \sigma_{-a}^{(-\lambda)}(x). \tag{4}$$

□

Theorem 4.8. *Subtraction is closed over pliant inequalities and the subtraction function can be given as*

$$\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)} = \sigma_{a_1-a_2}^{\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)}.$$

Proof. Let

$$\delta = \sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}$$

by using Lemma 2.1 and Lemma 4.7 we have

$$\begin{aligned}
\delta^{-1} &= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} - \left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1} = \\
&= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} + \left(-\left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1}\right) = \\
&= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} + \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right)^{-1} = \\
&= \left(\sigma_{a_1}^{(\lambda_1)} \oplus \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right)\right)^{-1},
\end{aligned}$$

therefore

$$\left(\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}\right) = \sigma_{a_1}^{(\lambda_1)} \oplus \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right). \tag{5}$$

Substituting Eq. 4 and Eq. 3 into Eq. 5 we get the desired result

$$\left(\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}\right) = \sigma_{a_1}^{(\lambda_1)} \oplus \sigma_{-a_2}^{(-\lambda_2)} = \sigma_{a_1-a_2}^{\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)}. \tag{6}$$

□

Note: The function does not exist in case of $\lambda_1 - \lambda_2 = 0$.

4.5 Multiplication and the n$^{\text{th}}$ power

Now let $\delta = \sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}$. By using Lemma 2.1 we get

$$\begin{aligned}
\delta^{-1}(z) &= \left[\left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1}(z)\right]\left[\left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1}(z)\right] = \\
&= \left[\frac{1}{-\lambda_1}\ln\left(\frac{1-z}{z}\right)+a_1\right]\left[\frac{1}{-\lambda_2}\ln\left(\frac{1-z}{z}\right)+a_2\right] = \\
&= \frac{1}{\lambda_1\lambda_2}\ln^2\left(\frac{1-z}{z}\right)+\left(\frac{a_2}{-\lambda_1}+\frac{a_1}{-\lambda_2}\right)\ln\left(\frac{1-z}{z}\right)+a_1a_2 \\
&= \frac{1}{\lambda_1\lambda_2}\ln^2\left(\frac{1-z}{z}\right)-\frac{\lambda_1a_1+\lambda_2a_2}{\lambda_1\lambda_2}\ln\left(\frac{1-z}{z}\right)+a_1a_2 \qquad (7)
\end{aligned}$$

Unfortunately this is not an inverse pliant inequality and it is not monotone. To obtain the roots of the function we set $\left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1}(z) = 0$ to get $z_0 = \frac{1}{1+e^{\lambda_1 a_1}}$ and we set $\left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1}(z) = 0$ to get $z_1 = \frac{1}{1+e^{\lambda_2 a_2}}$.

A complete analysis of δ^{-1} would require checking both the $\lambda_1\lambda_2 > 0$ and $\lambda_1\lambda_2 < 0$ cases. However by Criteria 4.4 we only need to examine the first case.

Let $\lambda_1\lambda_2 > 0$. In this case $\delta^{-1}(z) < 0$ when $z \in (min(z_0, z_1), max(z_0, z_1))$. By using the first derivative we get $z_{min} = \frac{1}{1+e^{\frac{1}{2}(\lambda_1a_1+\lambda_2a_2)}}$. Let us transform Eq. 7 to get z on the left side of the equation

$$\begin{aligned}
\frac{1}{\lambda_1\lambda_2}\ln^2\left(\frac{1-z}{z}\right)-\frac{\lambda_1a_1+\lambda_2a_2}{\lambda_1\lambda_2}\ln\left(\frac{1-z}{z}\right)+a_1a_2 &= x \\
\frac{1}{\lambda_1\lambda_2}\left(\ln\left(\frac{1-z}{z}\right)-\frac{1}{2}\left(\lambda_1a_1+\lambda_2a_2\right)\right)^2 &=
\end{aligned}$$

$$
\begin{aligned}
&= x - a_1 a_2 + \frac{(\lambda_1 a_1 + \lambda_2 a_2)^2}{4\lambda_1\lambda_2} \\
\left(\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)^2 &= \lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2 \\
\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2) &= \pm\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} \quad (8)
\end{aligned}
$$

We need to check two cases here. First, let $z \in (0, z_{min}]$. In this case $\delta^{-1}(z)$ is strictly monotone decreasing (thus has an inverse) and the left side of Eq. 8 is positive therefore

$$
\ln\left(\frac{1-z}{z}\right) - \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2) = \sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2}.
$$

From here we have

$$
\begin{aligned}
\ln\left(\frac{1-z}{z}\right) &= \sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2) \\
\frac{1-z}{z} &= e^{\left(\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)} \\
z &= \frac{1}{1 + e^{\left(\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)}}
\end{aligned}
$$

For $\left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}\right)(x)$ this gives us

$$
\delta'(x) = \left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}\right)(x) = \frac{1}{1 + e^{\left(\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)}}
$$

which is not a pliant inequality though rather similar. The domain of $\delta'(x)$ is $x \in [-\frac{1}{\lambda_1\lambda_2}\left(\frac{a_1\lambda_1 - a_2\lambda_2}{2}\right)^2, \infty)$.

For the second case we have $z \in [z_{min}, 1)$. Now $\delta^{-1}(z)$ is strictly monotone increasing (thus has an inverse) and the left side of Eq. 8 is negative. Now we have

$$
\begin{aligned}
\ln\left(\frac{1-z}{z}\right) &= -\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2) \\
\frac{1-z}{z} &= e^{\left(-\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)} \\
z &= \frac{1}{1 + e^{\left(-\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)}}
\end{aligned}
$$

For $\left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}\right)(x)$ we get

$$\delta''(x) = \left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}\right)(x) \quad = \quad \frac{1}{1 + e^{\left(-\sqrt{\lambda_1\lambda_2 x + \frac{1}{4}(\lambda_1 a_1 - \lambda_2 a_2)^2} + \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2)\right)}}$$

The domain of δ'' is $x \in [-\frac{1}{\lambda_1\lambda_2}\left(\frac{a_1\lambda_1 - a_2\lambda_2}{2}\right)^2, \infty)$.

Note that here λ_i and a_i play *the analogous roles* of m_i and a_i in additive pliant multiplication.

Functions σ' and σ'' shall be used if we need to multiply pliant inequalities from the positive range of the x axis. If $\lambda_1, \lambda_2 < 0$ then σ' should be used because it maps the positive part of $\sigma_{a_1}^{(\lambda_1)}$ and $\sigma_{a_2}^{(\lambda_2)}$. If $\lambda_1, \lambda_2 > 0$ then σ'' should be used.

Fig. 7 demonstrates multiplication of $\sigma_{a_1}^{(\lambda_1)}$ and $\sigma_{a_2}^{(\lambda_2)}$ pliant inequalities. The parameters of the operands were $\lambda_1 = 0.9, a_1 = 4$ and $\lambda_2 = 0.8, a_2 = 7$ respectively. The result is shown as the dotted curve.

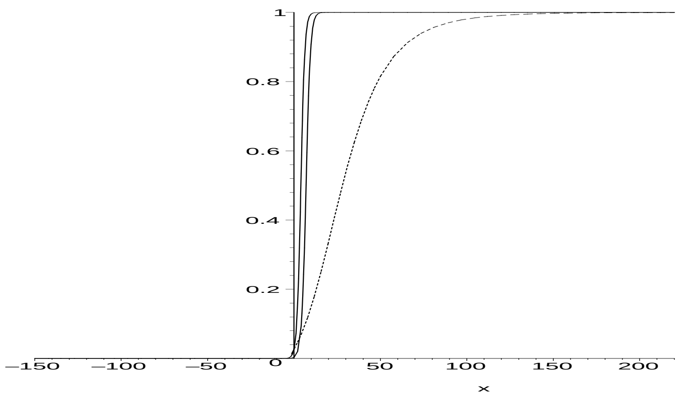

Figure 7: Multiplication of pliant inequalities.

The calculations show that multiplication is not closed for pliant inequalities. Nevertheless we can approximate the result well with a pliant inequality. We construct this function to take the value $\frac{1}{2}$ at $a_1 a_2$ and let the tangent here be the same as of the multiplication function. For the tangent we have $\frac{1}{4}\frac{1}{\frac{a_1}{\lambda_2} + \frac{a_2}{\lambda_1}}$, and our approximation function is

$$\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)} \approx \sigma_{a_1 a_2}^{\left(\frac{1}{4}\frac{1}{\frac{a_1}{\lambda_2} + \frac{a_2}{\lambda_1}}\right)}.$$

Fig. 8 shows the approximation of the multiplication in Fig. 7. The approximating pliant inequality is plotted as the dotted curve.

We can use the approximation function for *large* values of a_1, a_2 or large values of $|\lambda_1|, |\lambda_2|$. In this case the function values are small for $x < 0$ thus the approximation is better.

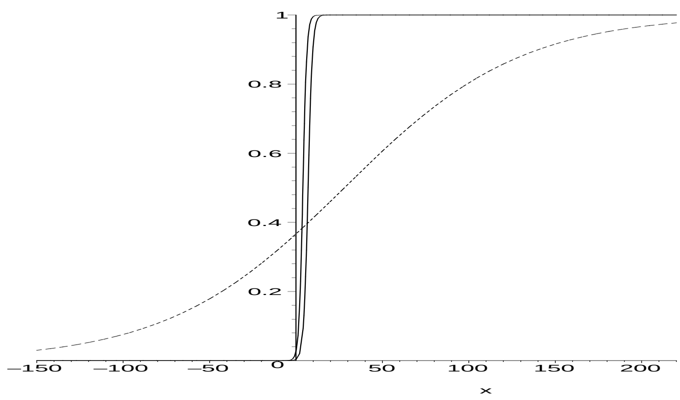

Figure 8: Approximation of pliant multiplication.

Let us calculate the n[th] power of $\sigma_a^{(\lambda)}$. This case

$$\delta^{-1}(z) = \left[\sigma_a^{(\lambda)}\right]^n = \left[\frac{1}{-\lambda}\ln\left(\frac{1-z}{z}\right) + a\right]^n$$

and then

$$\delta(x) = \frac{1}{1 + e^{-\lambda(\sqrt[n]{x} - a)}}.$$

We can approximate the power function with a pliant inequality similarly to the multiplication function. Let the approximating power function take $\frac{1}{2}$ in a^n and let the tangent be the same here as of the power function. For the tangent this gives $\frac{1}{4}\frac{\lambda}{na^{n-1}}$ and our approximation function is

$$\left(\sigma_a^{(\lambda)}\right)^n \approx \sigma_{a^n}^{\left(\frac{1}{4}\frac{\lambda}{na^{n-1}}\right)}.$$

4.6 Division

Let $\delta = \sigma_{a_1}^{(\lambda_1)} \oslash \sigma_{a_2}^{(\lambda_2)}$. By using Lemma 2.1 we get

$$\delta^{-1}(z) = \frac{\left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1}(z)}{\left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1}(z)} = \frac{\frac{1}{-\lambda_1}\ln\left(\frac{1-z}{z}\right) + a_1}{\frac{1}{-\lambda_2}\ln\left(\frac{1-z}{z}\right) + a_2}. \quad (9)$$

Note that the function is undefined when the divisor equals zero, i.e. $z_0 = \frac{1}{1+e^{(\lambda_2 a_2)}}$. The function $\delta^{-1}(z)$ is strictly monotone decreasing in $(0, z_0)$ and strictly monotone increasing in $(z_0, 1)$. The limit of the function in 0 and in 1 is λ_2/λ_1. This means that the function does not take any value twice thus has an inverse and its domain is all real numbers except λ_2/λ_1. Let us transform Eq. 9 to get z on the left side of the equation

$$\begin{aligned}
\frac{\frac{1}{-\lambda_1}\ln\left(\frac{1-z}{z}\right)+a_1}{\frac{1}{-\lambda_2}\ln\left(\frac{1-z}{z}\right)+a_2} &= x \\
\frac{1}{-\lambda_1}\ln\left(\frac{1-z}{z}\right)+a_1 &= xa_2-\frac{1}{\lambda_2}\ln\left(\frac{1-z}{z}\right)x \\
\frac{1}{-\lambda_1}\ln\left(\frac{1-z}{z}\right)+\frac{x}{\lambda_2}\ln\left(\frac{1-z}{z}\right) &= xa_2-a_1 \\
\ln\left(\frac{1-z}{z}\right)=\frac{xa_2-a_1}{\frac{\lambda_2-x\lambda_1}{-\lambda_1\lambda_2}} &= \frac{-\lambda_1\lambda_2}{\lambda_2-x\lambda_1}(xa_2-a_1) \\
\frac{1-z}{z} &= e^{\frac{-\lambda_1\lambda_2}{\lambda_2-x\lambda_1}(xa_2-a_1)} \\
z &= \frac{1}{1+e^{\frac{-\lambda_1\lambda_2}{\lambda_2-x\lambda_1}(xa_2-a_1)}}.
\end{aligned}$$

From here we have

$$\delta(x)=\left(\sigma_{a_1}^{(\lambda_1)}\oslash\sigma_{a_2}^{(\lambda_2)}\right)(x)=\frac{1}{1+e^{\frac{-\lambda_1\lambda_2}{\lambda_2-x\lambda_1}(xa_2-a_1)}}.$$

The result is not a pliant inequality. Fig. 9 presents the division of $\sigma_{a_1}^{(\lambda_1)}$ with $\sigma_{a_2}^{(\lambda_2)}$. The parameters were the same as in the multiplication example.

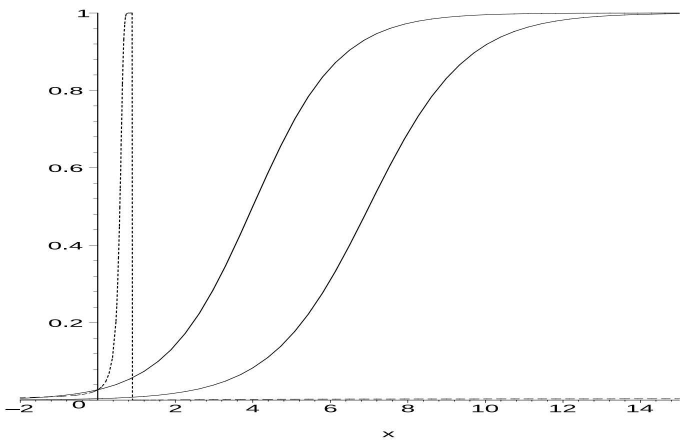

Figure 9: Division of pliant inequalities.

Based on the approximation of the multiplication function we can use the following pliant inequality to approximate division.

$$\sigma_{a_1}^{(\lambda_1)} \oslash \sigma_{a_2}^{(\lambda_2)} \approx \sigma_{\frac{a_1}{a_2}}^{\frac{a_2}{4\lambda_1 - \frac{a_1}{a_2 \lambda_2}}}.$$

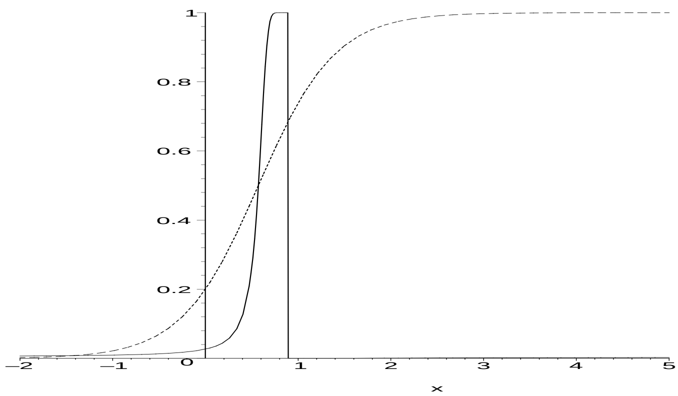

Figure 10: Approximation of pliant division.

4.7 Properties of Operations

Theorem 4.9. *Addition is commutative and associative over pliant inequalities.*

Proof. These properties can be easily seen from the construction of $\frac{1}{m}$ and a in Theorem 4.5. $\square$

Theorem 2.2 guarantees that multiplication over pliant inequalities is commutative, associative and distributive over addition. Now we prove these properties for the approximation of multiplication.

Theorem 4.10. *The approximation of multiplication over pliant inequalities is commutative, associative and distributive over addition.*

Proof. Commutativity

$$\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)} = \sigma_{a_1+a_2}^{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}} = \sigma_{a_2+a_1}^{\frac{\lambda_2\lambda_1}{a_2\lambda_2+a_1\lambda_1}} = \sigma_{a_2}^{(\lambda_2)} \otimes \sigma_{a_1}^{(\lambda_1)}$$

Associativity
Since

$$\begin{aligned}
\frac{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}\lambda_3}{a_1a_2\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}+a_3\lambda_3} &= \frac{\frac{\lambda_1\lambda_2\lambda_3}{a_1\lambda_1+a_2\lambda_2}}{\frac{a_1a_2\lambda_1\lambda_2+a_1a_3\lambda_1\lambda_3+a_2a_3\lambda_2\lambda_3}{a_1\lambda_1+a_2\lambda_2}} \\
&= \frac{\lambda_1\lambda_2\lambda_3}{a_1a_2\lambda_1\lambda_2+a_1a_3\lambda_1\lambda_3+a_2a_3\lambda_2\lambda_3} = \\
&= \frac{\frac{\lambda_1\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}}{\frac{a_1a_2\lambda_1\lambda_2+a_1a_3\lambda_1\lambda_3+a_2a_3\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}} = \\
&= \frac{\lambda_1\frac{\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}}{a_1\lambda_1+a_2a_3\frac{\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}},
\end{aligned}$$

we have

$$\begin{aligned}
(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}) \otimes \sigma_{a_3}^{(\lambda_3)} &= \sigma_{a_1a_2}^{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}} \otimes \sigma_{a_3}^{(\lambda_3)} = \\
&= \sigma_{(a_1a_2)a_3}^{\left(\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}\lambda_3\right)/\left(a_1a_2\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}+a_3\lambda_3\right)} = \\
&= \sigma_{a_1(a_2a_3)}^{\left(\lambda_1\frac{\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}\right)/\left(a_1\lambda_1+a_2a_3\frac{\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}\right)} = \\
&= \sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2a_3}^{\frac{\lambda_2\lambda_3}{a_2\lambda_2+a_3\lambda_3}} = \sigma_{a_1}^{(\lambda_1)} \otimes (\sigma_{a_2}^{(\lambda_2)} \otimes \sigma_{a_3}^{(\lambda_3)}).
\end{aligned}$$

Distributivity
We shall use the following equality

$$
\begin{aligned}
\frac{\lambda_1 \frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}}{a_1\lambda_1 + (a_2+a_3)\frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}} &= \frac{\frac{1}{\lambda_2+\lambda_3}}{\frac{1}{\lambda_2+\lambda_3}} \frac{\lambda_1\lambda_2\lambda_3}{a_1\lambda_1(\lambda_2+\lambda_3) + (a_2+a_3)\lambda_2\lambda_3} = \\
&= \frac{\lambda_1\lambda_2\lambda_1\lambda_3}{a_1\lambda_1^2\lambda_2 + a_1\lambda_1^2\lambda_3 + a_2\lambda_1\lambda_2\lambda_3 + a_3\lambda_1\lambda_2\lambda_3} = \\
&= \frac{\frac{1}{(a_1\lambda_1+a_2\lambda_2)}\frac{1}{(a_1\lambda_1+a_3\lambda_3)}}{\frac{1}{(a_1\lambda_1+a_2\lambda_2)}\frac{1}{(a_1\lambda_1+a_3\lambda_3)}} \times \\
&\quad \times \frac{\lambda_1\lambda_2\lambda_1\lambda_3}{((a_1\lambda_1+a_3\lambda_3)\lambda_1\lambda_2 + (a_1\lambda_1+a_2\lambda_2)\lambda_1\lambda_3)} = \\
&= \frac{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}\frac{\lambda_1\lambda_3}{a_1\lambda_1+a_3\lambda_3}}{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2} + \frac{\lambda_1\lambda_3}{a_1\lambda_1+a_3\lambda_3}}.
\end{aligned}
$$

From here we have

$$
\begin{aligned}
\sigma_{a_1}^{(\lambda_1)} \otimes (\sigma_{a_2}^{(\lambda_2)} \oplus \sigma_{a_3}^{(\lambda_3)}) &= \sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2+a_3}^{\frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}} = \sigma_{a_1(a_2+a_3)}^{\frac{\lambda_1\frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}}{a_1\lambda_1+(a_2+a_3)\frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}}} = \\
&= \sigma_{a_1a_2+a_1a_3}^{\frac{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}\frac{\lambda_1\lambda_3}{a_1\lambda_1+a_3\lambda_3}}{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}+\frac{\lambda_1\lambda_3}{a_1\lambda_1+a_3\lambda_3}}} = \\
&= \sigma_{a_1+a_2}^{\frac{\lambda_1\lambda_2}{a_1\lambda_1+a_2\lambda_2}} \oplus \sigma_{a_1+a_3}^{\frac{\lambda_1\lambda_3}{a_1\lambda_1+a_3\lambda_3}} = \\
&= \left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_2}^{(\lambda_2)}\right) \oplus \left(\sigma_{a_1}^{(\lambda_1)} \otimes \sigma_{a_3}^{(\lambda_3)}\right).
\end{aligned}
$$

□

5 Pliant Numbers and Operator Families

Pliant numbers can be decomposed to left hand side and right hand side fuzzy inequalities. In the previous sections we have used this decomposition to investigate two classes of suitable functions and their related arithmetic operations. After we have calculated the result of an arithmetic operation we need to combine the left hand side and the right hand side results using a fuzzy conjunction operator.

In this section we examine some classic fuzzy conjunction operators to investigate which ones are the most suitable for lines and pliant inequalities.

5.1 Pliant Numbers and linear function

Let us assume we would like to calculate the sum of two pliant numbers represented with lines. To this end we add the left lines and the right lines separately. Before applying conjunction we need to cut the results so they map to the $[0,1]$ interval. This can be achieved using the cut function defined as

$$[x] \quad = \quad \min\left(\max(x,0),1\right).$$

Fig. 11 presents a sample left and right line with the cut function applied.

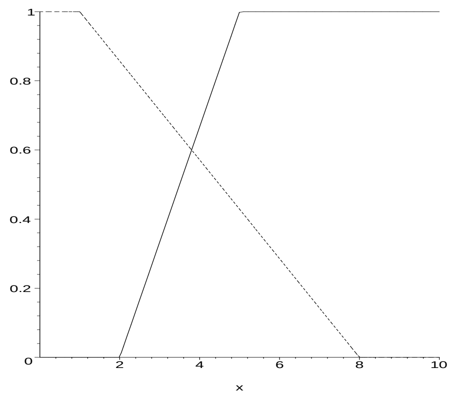

Figure 11: Cut function applied to lines.

The final step of addition is to construct the pliant number using a conjunction operator. Two cases should be considered here depending on how the two left and right hand side functions overlap.

If the two functions do not intersect each other before they reach one then any conjunction operator can be used since any $c(x,y)$ conjunction satisfies $c(1,x) = x$ and $c(x,1) = x$ which guarantees that the lines remain intact.

Now if the two functions intersect each other before they reach one then we need to make sure that the conjunction operator does not distort the lines. Here we can also distinguish two cases depending whether the left line and the right line has the same absolute tangent or not. The left side of Fig. 12 shows a left line with $m = \frac{1}{4}, a = 4$ and a right line with $m = -\frac{1}{4}, a = 6$ parameters. The right side of the figure shows the same left line, but a different right line with parameters $m = -\frac{1}{6}, a = 6$.

Let us now create a pliant number from these two linesets using the algebraic product $(a \cdot b)$ as shown in Fig. 13.
As it can be seen from the figure the algebraic product conjunction distorts the lines around the point of their intersection.

Let us now apply the classic minimum conjunction.
Fig. 14 shows that the minimum conjunction simply cuts everything above the intersection point but leaves the remaining line segments undistorted.

Let us see what happens if we use the Lukasiewicz or bounded operator. $([x + y - 1])$.

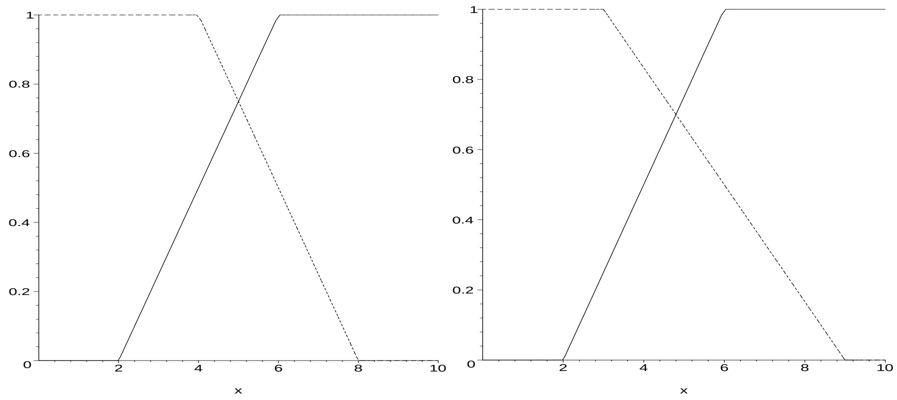

Figure 12: Lines with same and different absolute tangents.

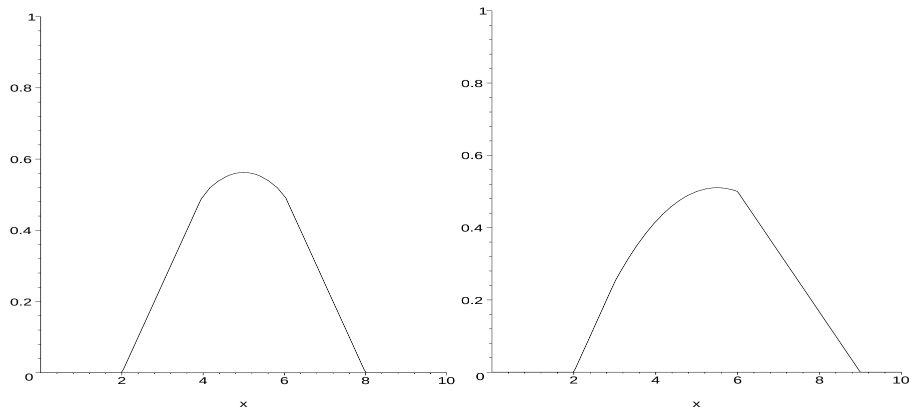

Figure 13: Product conjunction with same and different absolute tangent lines.

As it can be seen from Fig. 15 the results are different from each other. On the left side of the figure the lines had the same absolute tangent which resulted in a trapezoid form while on the right side the lines had different absolute tangents which resulted in a cut-off point on the line with a greater absolute tangent value.

The results from the minimum and the Lukasiewicz conjunction resulted in lines which makes them a good choice for creating pliant numbers from additive pliant.

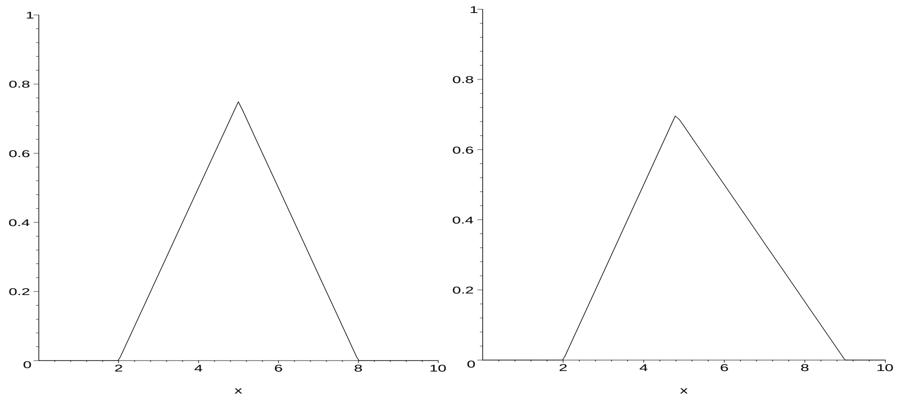

Figure 14: Minimum conjunction with same and different absolute tangent lines.

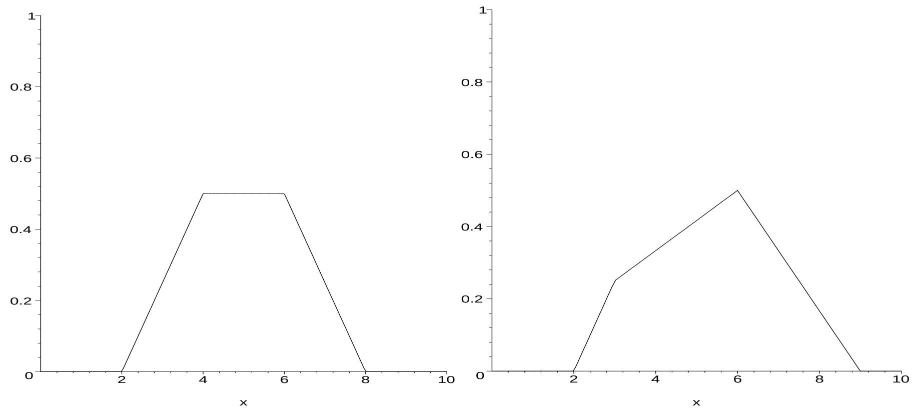

Figure 15: Bounded conjunction with same and different absolute tangent lines.

5.2 Pliant Numbers with Pliant Inequalities

Let us assume we would like to calculate the sum of two pliant numbers represented with pliant inequalities. This case we need to add two left hand side pliant inequalities together and add two right hand side pliant inequalities together. We can depict the operation using the following notation.

$$\begin{bmatrix} \lambda_l^1 & \lambda_r^1 \\ a_l^1 & a_r^1 \end{bmatrix} \oplus \begin{bmatrix} \lambda_l^2 & \lambda_r^2 \\ a_l^2 & a_r^2 \end{bmatrix} = \begin{bmatrix} \lambda_l & \lambda_r \\ a_l & a_r \end{bmatrix}$$

where

$$\frac{1}{\lambda_l} = \frac{1}{\lambda_l^1} + \frac{1}{\lambda_l^2}, \qquad \frac{1}{\lambda_r} = \frac{1}{\lambda_r^1} + \frac{1}{\lambda_r^2} \quad \text{and} \quad a_l = a_l^1 + a_l^2, \qquad a_r = a_r^1 + a_r^2.$$

The subscripts denote the left and the right hand side operands respectively, superscripts denote the two pliant numbers and the results are denoted without superscripts. Theorem 4.5 defines the relationship between the parameters.

The final step of the addition is to construct the pliant number using a conjunction. Two cases should be considered here depending on how the two left and right hand side functions overlap.

First if the two functions do not intersect each other before they approach one then any conjunction operator can be used since any $c(x, y)$ conjunction satisfies $c(1, x) = x$ and $c(x, 1) = x$ which guarantees that the shape of the curves will not be distorted. Fig. 16 presents this situation.

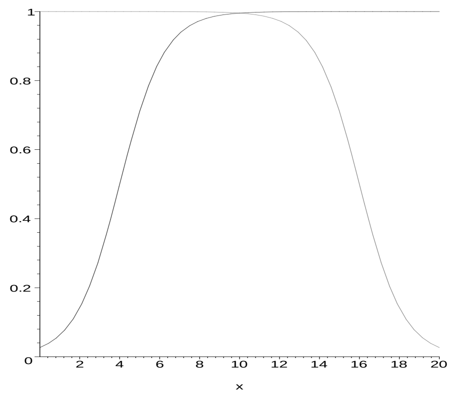

Figure 16: Two pliant inequalities.

Now if the two functions intersect each other before they approach one then we need to make sure that the conjunction operator does not distort the shape of the curves. This is shown in the left side of Fig. 17.

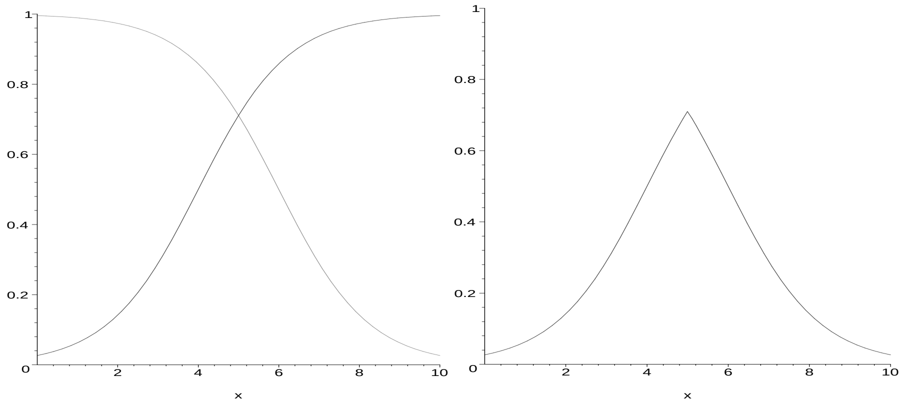

Figure 17: Two pliant inequalities, conjunction with $min(x, y)$.

The right side of Fig. 17 shows the result of using the $min(x, y)$ conjunction. As it can be seen from the figure this operator cuts the functions sharply. Fig. 18 shows the result of conjunctions using the algebraic operator $(a \cdot b)$ and the Lukasiewicz operator $([x + y - 1])$.

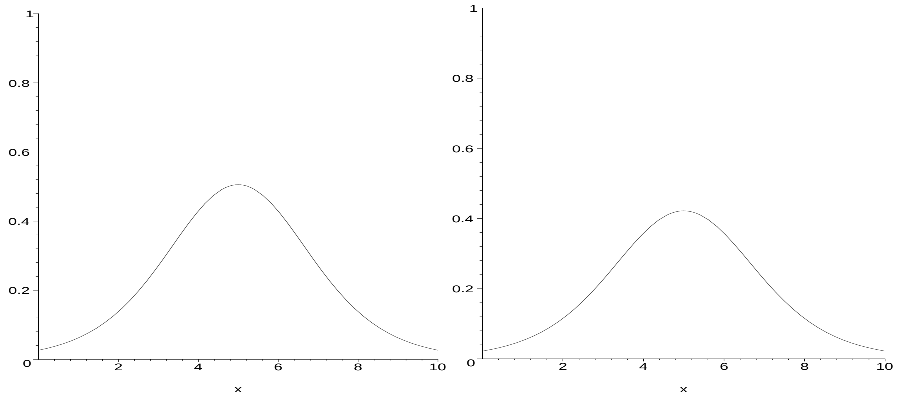

Figure 18: Conjunction with algebraic product and Lukasiewicz operator.

While these conjunctions result in a smooth curve, they squash the operands along the y-axis. The averaging Dombi operator is given as

$$\bar{c}(x, y) = \frac{1}{1 + \frac{1}{2}\left(\frac{1-x}{x} + \frac{1-y}{y}\right)}.$$

Let us now use the averaging Dombi operator to construct the pliant number.

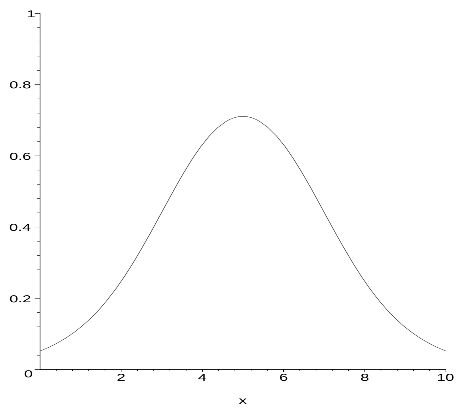

Figure 19: Conjunction with the Dombi operator.

This case the result is a smooth curve that reaches its maximum value at the intersection of the curves. The averaging Dombi operator also retains the parameter values of the two functions as the following calculations show.

$$\bar{c}\left(\sigma_{a_l}^{(\lambda_l)}, \sigma_{a_r}^{(\lambda_r)}\right) = \frac{1}{1+\frac{1}{2}\left(\frac{1-\sigma_{a_l}^{(\lambda_l)}}{\sigma_{a_l}^{(\lambda_l)}}+\frac{1-\sigma_{a_r}^{(\lambda_r)}}{\sigma_{a_r}^{(\lambda_r)}}\right)} = \frac{1}{1+\frac{1}{2}\left(e^{-\lambda_l(x-a_l)}+e^{-\lambda_r(x-a_r)}\right)}.$$

This enables us to easily decompose the result for further arithmetic operations. These properties makes the Dombi operator a good choice for constructing pliant numbers based on pliant inequalities.

Acknowledgements

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