

Type-2 implications on non-interactive fuzzy truth values

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Abstract

In this paper we consider algebras of fuzzy truth values equipped with an implication operation. Generalizing the definition of type-1 fuzzy implications, we discuss extended fuzzy S-implications and coimplications and also extended residual implications and coimplications, in particular to the extended residual of minimum. We study the algebraic properties of these operations and their relationships to extended t-norms and t-conorms in particular to meet and join. These investigations are intended to provide a theoretical background for type-2 approximate reasoning applications.

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1. Introduction

Fuzzy logic in narrow sense is a generalization of classical two-valued logic, it considers a range of truth values, usually the unit interval. Although fuzzy logic became the “language” of vague propositions, its $[0, 1]$ -valued truth values are still precise. In recent years, research related to type-2 fuzzy logic has become even more active than ever as they seem to provide a better framework for the “computing with words” paradigm than classical fuzzy sets [11]. Type-2 fuzzy logic takes the generalization a step further by considering truth values that are themselves fuzzy. This means that every truth value (i.e. every element of $[0, 1]$) has a fuzzy membership degree (which is again an element of $[0, 1]$). This mapping from the unit interval to itself is the truth value, hence its name fuzzy truth value.

The recent popularity of type-2 fuzzy logic mainly stem from the works of Mendel et al. [10,12,13]. Besides the numerous papers with applications of type-2 fuzzy sets, there are many contributions from the theoretical line of research, too. The theory of type-2 fuzzy sets was established by Zadeh [23], Mizomoto and Tanaka [14,15], Dubois and Prade [4,5], and Nieminen [16]. Recent publications by Walker and Walker [22] and Starczewski [19,18] unfold the rich algebraic structure of fuzzy truth values. These papers consider type-2 t-norms and t-conorms on fuzzy truth values, either in general or by restricting the set of fuzzy truth values to—for example—normal, convex, triangular, trapezoidal or bell-shaped functions.

As the basic building blocks of any inference process, fuzzy implications have always been in the mainstream research. The study of (type-1) fuzzy implications [6–8,17,20,21] provided the basis for the theory and practice of approximate reasoning. Although implicative operations (such as implications and coimplications) were investigated on interval-valued fuzzy sets (which are special type-2 fuzzy sets with intervals as truth values), a general discussion

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of type-2 implicative operations did not exist. In this paper we study type-2 implicative operations on non-interactive fuzzy truth values. The use of type-2 implicative operations is not straightforward, for example not all properties of a type-1 fuzzy implication apply to its extension.

This paper is organized as follows. Section 2 contains elementary definitions and results in recent literature. Section 3 investigates the properties of type-2 S-implications and coimplications in general, and their relationships to type-2 t-norms and t-conorms. Because of the special importance of the extended minimum and maximum (i.e. meet and join), S-implications and coimplication formed by these operations are discussed separately. Extended residual implications, in particular the extended residual of \wedge is studied in Section 4, with an in detail examination of subalgebras of fuzzy truth values equipped with it. Finally, Section 5 contains some concluding remarks.

2. Preliminaries

Let the real unit interval be denoted by \mathcal{I} .

Definition 1. A t-norm is a binary operation $\Delta : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ that is commutative, associative, increasing in each variable, and has unit element 1.

Definition 2. A t-conorm is a binary operation $\nabla : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ that is commutative, associative, increasing in each variable, and has unit element 0.

Definition 3. Fuzzy implications are two-place functions $\triangleright : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ which fulfill the boundary conditions according to the Boolean implication, and antitone in the first and monotone in the second argument.

Definition 4. A fuzzy implication \triangleright is the residual implication associated with a t-norm Δ if

$$x \triangleright y = \bigvee_{(x \Delta z) \leq y} z.$$

Definition 5. A strong negation $'$ is an involutive order reversing automorphism of \mathcal{I} .

Definition 6. Fuzzy coimplications \triangleleft are dual to fuzzy implications by a strong negation according to

$$x \triangleleft y = (x' \triangleright y')'.$$

A fuzzy coimplication \triangleleft is the residual coimplication of a t-conorm ∇ if

$$x \triangleleft y = \bigwedge_{(x \nabla z) \geq y} z.$$

Throughout this paper we will consider continuous t-norms, t-conorms and negations.

Definition 7. Fuzzy truth values are mappings of \mathcal{I} onto itself. The set of fuzzy truth values is denoted by \mathcal{F} .

We remark that this definition of fuzzy truth values can be generalized in many ways (for example by considering a lattice instead of \mathcal{I}), but in this paper we will use this one.

For a fuzzy truth value f let

$$f^R(x) = \bigvee_{y \geq x} f(y) \quad \text{and} \quad f^L(x) = \bigvee_{y \leq x} f(y).$$

Note that $(f^L)^R = (f^R)^L = f^{LR}$ is a constant function which takes the supremum of f everywhere. The following hold for all $f, g \in \mathcal{F}$ (\leq is a pointwise relation):

(1) $f \leq f^R; f \leq f^L.$

- (2) $(f \wedge g)^R \leq f^R \wedge g^R; (f \wedge g)^L \leq f^L \wedge g^L.$
- (3) $(f \vee g)^R = f^R \vee g^R; (f \vee g)^L = f^L \vee g^L.$

Definition 8. A fuzzy truth value $f \in \mathcal{F}$ is

- (1) left-maximal (resp. right-maximal) if $f^L = f^{LR}$ (resp. $f^R = f^{LR}$);
- (2) normal if f^{LR} is the constant function 1. The set of normal fuzzy truth values is denoted by \mathcal{F}_N ;
- (3) convex if for all $x \leq y \leq z, f(x) \wedge f(z) \leq f(y)$, or equivalently if $f = f^L \wedge f^R$ (see [22]). The set of convex fuzzy truth values is denoted by \mathcal{F}_C ;
- (4) an interval if it is the characteristic function of a closed subinterval of \mathcal{I} . The set of interval fuzzy truth values is denoted by \mathcal{F}_I ;
- (5) monotone increasing if and only if $f^L = f$, and monotone decreasing if and only if $f^R = f$. These sets of fuzzy truth values will be denoted by \mathcal{F}^+ and \mathcal{F}^- , respectively.

Note that there are further conditions of normality which are equivalent to the definition. For example, $f \in \mathcal{F}_N$ if and only if $f^R(0) = 1$ (or $f^L(1) = 1$), because $f^R(0)$ is the supremum of f over the real unit interval.

Two special fuzzy truth values are the following:

$$\mathbf{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{1}(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

According to Zadeh’s extension principle, a two-place function $\circ : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$ can be extended to $\bullet : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ by the convolution of \circ with respect to \wedge and \vee . Let $f, g \in \mathcal{F}$, then

$$(f \bullet g)(z) = \bigvee_{z=x \circ y} (f(x) \wedge g(y)).$$

Throughout this paper we suppose non-interactivity, i.e. \wedge is not generalized to a t-norm.

A unary function, such as a strong negation $'$ on \mathcal{I} extended on the set of fuzzy truth values has the following definition:

$$f^*(x) = \bigvee_{x=y'} f(y) = f(x').$$

In particular, if \circ is a t-norm \triangle or a t-conorm ∇ , its extension is called a type-2 t-norm or t-conorm. We have the following definitions for type-2 t-norms and t-conorms.

Definition 9. Let \triangle and ∇ be a t-norm and a t-conorm, then their extensions \blacktriangle and \blacktriangledown are defined as follows:

$$(f \blacktriangle g)(z) = \bigvee_{z=x \triangle y} (f(x) \wedge g(y)),$$

$$(f \blacktriangledown g)(z) = \bigvee_{z=x \nabla y} (f(x) \wedge g(y)).$$

The following properties hold for these operations (for a more comprehensive list see [22]):

- (1) Both are commutative and associative.
- (2) $(f \blacktriangle g)^L = f^L \blacktriangle g^L; (f \blacktriangledown g)^L = f^L \blacktriangledown g^L.$
- (3) $f \blacktriangle \mathbf{1} = f; f \blacktriangledown \mathbf{0} = f$ for all $f \in \mathcal{F}$.
- (4) $\mathbf{1} \blacktriangledown \mathbf{1} = \mathbf{1}; \mathbf{0} \blacktriangle \mathbf{0} = \mathbf{0}.$

Here we prove only the first equality of item 4. Let $z = 1$, then

$$(\mathbf{1} \blacktriangledown \mathbf{1})(1) = \bigvee_{1=x \nabla y} (\mathbf{1}(x) \wedge \mathbf{1}(y)) = (\mathbf{1}(1) \wedge \mathbf{1}(1)) \vee \bigvee_{1=x \nabla y} (\mathbf{1}(x) \wedge \mathbf{1}(y)),$$

which clearly equals to 1, since $\mathbf{1}(1) \wedge \mathbf{1}(1) = 1$. For all $z < 1$,

$$(\mathbf{1}\blacktriangledown\mathbf{1})(z) = \bigvee_{1 > z = x \blacktriangledown y} (\mathbf{1}(x) \wedge \mathbf{1}(y)).$$

This subset of (x, y) pairs clearly does not contain $(1, 1)$ (because it would imply $z = 1$), so x or y is strictly less than 1, i.e. $\mathbf{1}(x)$ or $\mathbf{1}(y)$ is zero, which implies that their minimum is always 0.

The extended minimum and maximum (usually referred as meet and join) are fundamental operations on fuzzy truth values. These operations and their pointwise expressions are

$$(f \sqcap g)(z) = \bigvee_{z = x \wedge y} (f(x) \wedge g(y)) = ((f \wedge g^R) \vee (f^R \wedge g))(z),$$

$$(f \sqcup g)(z) = \bigvee_{z = x \vee y} (f(x) \wedge g(y)) = ((f \wedge g^L) \vee (f^L \wedge g))(z).$$

These operations define two partial orders \sqsubseteq and \preceq on \mathcal{F} . In particular, let

$$f \sqsubseteq g \quad \text{if and only if} \quad f \sqcap g = f,$$

$$f \preceq g \quad \text{if and only if} \quad f \sqcup g = g.$$

On the basis of the above, in this paper we will consider the algebra $\mathbf{F} = (\mathcal{F}, \sqcap, \sqcup, \blacktriangle, \blacktriangledown, *, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$ equipped with an implicative operation. This algebra is very general, it has many subalgebras with interesting and important properties. The subalgebra $\mathbf{F}_{CN} = (\mathcal{F}_{CN}, \sqcap, \sqcup, \blacktriangle, \blacktriangledown, *, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$ of \mathbf{F} i.e. the algebra of convex normal fuzzy truth values is of special importance. In \mathbf{F}_{CN} , the distributive and absorption laws hold for \sqcap and \sqcup (see [15]), i.e. for all $f, g, h \in \mathbf{F}_{CN}$,

$$f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h), \quad f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h),$$

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = f.$$

As a consequence, in \mathbf{F}_{CN} the partial orders \sqsubseteq and \preceq coincide, it is a bounded maximal lattice in \mathbf{F} (maximal among lattices containing an isomorph subalgebra to \mathcal{I}), and a De Morgan algebra. It is also complete, i.e. the operations \sqcap and \sqcup can be naturally extended to infinite operands since they are associative.

Fuzzy implications are defined on the algebra $\mathbf{I} = (\mathcal{I}, \wedge, \vee, \leq, 0, 1)$. We define type-2 fuzzy implications analogously to their type-1 counterparts. The underlying set of truth values is generalized from \mathcal{I} to a subset of \mathcal{F} , and since it may not be a lattice, the two partial orders defined by \sqcap and \sqcup are considered instead of \leq .

Definition 10. Let $\mathbf{A} = (\mathcal{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, where $\mathcal{A} \subseteq \mathcal{F}$. A function $\bullet : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is called a type-2 fuzzy implication over \mathbf{A} if and only if it satisfies the boundary conditions

$$\mathbf{0} \bullet \mathbf{0} = \mathbf{0} \bullet \mathbf{1} = \mathbf{1} \bullet \mathbf{1} = \mathbf{1}; \quad \mathbf{1} \bullet \mathbf{0} = \mathbf{0}$$

and it is antitone in the first and monotone in the second argument w.r.t. at least one of the partial orders \sqsubseteq or \preceq .

We make a careful distinction between extended fuzzy implications and type-2 fuzzy implications. For example, a fuzzy implication extended to \mathcal{F} is obviously an extended fuzzy implication, but it is a type-2 fuzzy implication (on a subset of \mathcal{F}) only if it satisfies the above conditions.

3. Extended S-implications and S-coimplications

S-implications are formed by a t-conorm \blacktriangledown and a strong negation $'$ according to the formula $x' \blacktriangledown y$. S-coimplications are dual to S-implications, and are defined as $x' \blacktriangle y$. The extensions of these operations are as follows:

$$(f \blacktriangleright g)(z) = \bigvee_{z = x' \blacktriangledown y} (f(x) \wedge g(y)) = (f^* \blacktriangledown g)(z),$$

$$(f \blacktriangleleft g)(z) = \bigvee_{z = x' \blacktriangle y} (f(x) \wedge g(y)) = (f^* \blacktriangle g)(z).$$

We assume that the underlying negation $'$, t-conorm ∇ and t-norm Δ of the operations \blacktriangleright and \blacktriangleleft are continuous. Since the operation \blacktriangleleft is dual to \blacktriangleright , i.e. $f \blacktriangleleft g = (f^* \blacktriangleright g^*)^*$ for all $f, g \in \mathcal{F}$, thus any statement involving \blacktriangleright has its dual with \blacktriangleleft . From now on, we omit the proofs of dual statements, these easily follow by duality.

Proposition 11. *The operations \blacktriangleright and \blacktriangleleft are closed on \mathcal{F}_C .*

Proof. The preservation of the convexity of fuzzy intervals over the real line was proved by e.g. [2] for any extended continuous function. Any fuzzy truth value can be naturally extended to the real line for example by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Since this extension does not affect the convexity of f , the statement follows. \square

Proposition 12. *The following hold for all $g, h \in \mathcal{F}$ if and only if f is convex:*

- (1) $(g \sqcap h) \blacktriangleright f = (g \blacktriangleright f) \sqcup (h \blacktriangleright f)$, $f \blacktriangleright (g \sqcap h) = (f \blacktriangleright g) \sqcap (f \blacktriangleright h)$,
- (2) $(g \sqcup h) \blacktriangleright f = (g \blacktriangleright f) \sqcap (h \blacktriangleright f)$, $f \blacktriangleright (g \sqcup h) = (f \blacktriangleright g) \sqcup (f \blacktriangleright h)$.

Proof. Straightforward from the distributivity of \blacktriangledown over \sqcap and \sqcup (see [22]) and the De Morgan law between \sqcap , \sqcup and $*$. \square

Note that similar laws apply to type-2 S-coimplications, too, because \blacktriangle also distributes over \sqcap and \sqcup by the same conditions.

Proposition 13. *The operations \blacktriangleright and \blacktriangleleft are closed on \mathcal{F}_N . Moreover, $f \blacktriangleright g$ and $f \blacktriangleleft g$ are normal if and only if $f, g \in \mathcal{F}_N$.*

Proof. We prove that \blacktriangledown is closed on \mathcal{F}_N , then the first statement follows from $f \blacktriangleright g = f^* \blacktriangledown g$, since $*$ preserves normality. Suppose $f, g \in \mathcal{F}_N$, i.e. $f^L(1) = g^L(1) = 1$. Then,

$$\begin{aligned} (f \blacktriangledown g)^L(1) &= (f^L \blacktriangledown g^L)(1) = \bigvee_{1=x \nabla y} (f^L(x) \wedge g^L(y)) \\ &= (f^L(1) \wedge g^L(1)) \vee \bigvee_{\substack{1=x \nabla y \\ x < 1 \text{ or } y < 1}} (f^L(x) \wedge g^L(y)) = 1, \end{aligned}$$

thus $f \blacktriangledown g$ is normal.

Now suppose $f \blacktriangledown g \in \mathcal{F}_N$. We have

$$1 = (f \blacktriangledown g)^R(0) = (f^R \blacktriangledown g^R)(0) = \bigvee_{0=x \nabla y} (f^R(x) \wedge g^R(y)) = f^R(0) \wedge g^R(0).$$

Note that in the last step the following property of t-conorms was used: $x \nabla y = 0$ if and only if $x = 0$ and $y = 0$. The above equality implies that $f^R(0) = g^R(0) = 1$, thus $f, g \in \mathcal{F}_N$.

Since the extended negation does not affect normality, the statement for \blacktriangleleft follow by duality. \square

Two important properties follow from the definition of type-1 fuzzy implications. For all fuzzy implications \triangleright ,

$$0 \triangleright x = 1 \quad \text{and} \quad x \triangleright 1 = 1 \quad \forall x \in [0, 1].$$

These properties can be naturally “extended” to fuzzy truth values:

$$0 \blacktriangleright f = \mathbf{1} \quad \text{and} \quad f \blacktriangleright \mathbf{1} = \mathbf{1} \quad \forall f \in \mathbf{A}$$

for a subalgebra \mathbf{A} of \mathbf{F} . The following proposition shows that these properties hold (and so it is reasonable to use \blacktriangleright on a subalgebra of \mathbf{A}) if and only if $\mathbf{A} \subseteq \mathbf{F}_N$.

Proposition 14. *The equations*

$$\mathbf{0} \blacktriangleright f = \mathbf{1} \quad \text{and} \quad f \blacktriangleright \mathbf{1} = \mathbf{1}$$

hold if and only if $f \in \mathcal{F}_N$.

Proof. We provide a proof only for the first equality, the other one can be proved similarly due to the duality between the utilized properties of (type-1) t-norms and t-conorms.

Let $f \in \mathcal{F}$, we have $\mathbf{0} \blacktriangleright f = \mathbf{1} \blacktriangledown f$, and by definition

$$(\mathbf{1} \blacktriangledown f)(z) = \bigvee_{z=x \blacktriangledown y} (\mathbf{1}(x) \wedge f(y)).$$

For a given $z < 1$, the minimum inside the sup is 0 whenever $x < 1$, since then $\mathbf{1}(x) = 0$. Since $x = 1$ would imply $z = 1$, the minimum is zero for every possible (x, y) pairs, hence $(\mathbf{0} \blacktriangleright f)(z) = 0$ for all $z < 1$.

In case $z = 1$, our claim is that

$$(\mathbf{1} \blacktriangledown f)(1) = \bigvee_{1=x \blacktriangledown y} (\mathbf{1}(x) \wedge f(y)) = 1.$$

Note that $x = 1$ is necessary, because otherwise $\mathbf{1}(x) = 0$, and the sup would never reach 1. So we have

$$\bigvee_{1=1 \blacktriangledown y} (\mathbf{1}(1) \wedge f(y)) = \bigvee_{y \in [0,1]} f(y),$$

which equals to 1 if and only if f is normal. \square

Clearly, by duality

$$\mathbf{1} \blacktriangleleft f = \mathbf{0} \quad \text{and} \quad f \blacktriangleleft \mathbf{0} = \mathbf{0} \quad \text{if and only if} \quad f \in \mathcal{F}_N.$$

The main result of this section shows that extended S-implications are type-2 fuzzy S-implications only on \mathcal{F}_{CN} .

Theorem 15. *The operation \blacktriangleright is a type-2 fuzzy implication over $\mathbf{A} \subseteq \mathbf{F}$ if and only if \mathbf{A} is a subalgebra of the algebra of convex normal functions \mathbf{F}_{CN} .*

Proof. Let $f, g, h \in \mathbf{A}$. By definition $f \blacktriangleright g = f^* \blacktriangledown g$. The boundary conditions hold since $f \blacktriangledown \mathbf{0} = f$, for all $f \in \mathcal{F}$, and $\mathbf{1} \blacktriangledown \mathbf{1} = \mathbf{1}$.

The sufficiency of the condition can be easily proved by Proposition 12 and considering that for convex normal fuzzy truth values the two partial orders coincide. Necessity can be proved as follows.

The operation \blacktriangleright is monotone in the second argument if $g \sqsubseteq h$ implies $f \blacktriangleright g \sqsubseteq f \blacktriangleright h$ (a similar argument follows from the use of \preceq) which is equivalent to

$$(f \blacktriangleright g) \sqcap (f \blacktriangleright h) = f \blacktriangleright g.$$

Now, $g \sqsubseteq h$, i.e. $g \sqcap h = g$, thus we have

$$(f \blacktriangleright g) \sqcap (f \blacktriangleright h) = f \blacktriangleright (g \sqcap h).$$

By Proposition 12 this holds if and only if f is convex, thus $\mathbf{A} \subseteq \mathbf{F}_C$.

The operation \blacktriangleright is antitone in the first argument if $g \sqsubseteq f$ implies $f \blacktriangleright h \sqsubseteq g \blacktriangleright h$ (again, a similar argument follows from the use of \preceq) which is equivalent to

$$(f \blacktriangleright h) \sqcap (g \blacktriangleright h) = f \blacktriangleright h.$$

Now, $g \sqsubseteq f$, i.e. $g \sqcap f = g$, thus we have

$$(f \blacktriangleright h) \sqcap ((f \sqcap g) \blacktriangleright h) = f \blacktriangleright h.$$

We may assume convexity, thus by Proposition 12, we have

$$(f \blacktriangleright h) \sqcap ((f \blacktriangleright h) \sqcup (g \blacktriangleright h)) = f \blacktriangleright h.$$

This is the absorption law, which holds if and only if $f \blacktriangleright h$ is convex and $g \blacktriangleright h$ is normal [22]. So, by Proposition 13, $g, h \in \mathcal{F}_N$ and thus $\mathbf{A} \subseteq \mathbf{F}_{CN}$. \square

Clearly, by duality the operation \blacktriangleleft is a type-2 fuzzy coimplication over $\mathbf{A} \subseteq \mathbf{F}$ if and only if \mathbf{A} is a subalgebra of the algebra of convex normal functions \mathbf{F}_{CN} .

Further properties of extended S-implications are summarized as follows.

Proposition 16. *The following hold for all $f, g, h \in \mathbf{F}$:*

- (1) $\mathbf{1} \blacktriangleright f = f$.
- (2) $f \blacktriangleright (g \blacktriangleright h) = g \blacktriangleright (f \blacktriangleright h)$.
- (3) $f \blacktriangleright g = g^* \blacktriangleright f^*$.
- (4) $g \sqsubseteq f \blacktriangleright g$ where $f, g \in \mathbf{F}_{CN}$.

Proof. Let $f, g, h \in \mathbf{F}$:

- (1) $\mathbf{1} \blacktriangleright f = (\mathbf{1}^*) \blacktriangledown f = \mathbf{0} \blacktriangledown f = f$.
- (2) $f \blacktriangleright (g \blacktriangleright h) = f^* \blacktriangledown (g^* \blacktriangledown h) = f^* \blacktriangledown g^* \blacktriangledown h = g^* \blacktriangledown (f^* \blacktriangledown h) = g \blacktriangleright (f \blacktriangleright h)$.
- (3) $g^* \blacktriangleright f^* = (g^*)^* \blacktriangledown f^* = f^* \blacktriangledown g = f \blacktriangleright g$.
- (4) To prove $g \sqsubseteq f \blacktriangleright g$ on \mathbf{F}_{CN} consider that

$$g \sqsubseteq f \blacktriangleright g \quad \text{iff} \quad g \sqcap (f \blacktriangleright g) = g$$

and

$$g \sqcap (f \blacktriangleright g) = (\mathbf{1} \blacktriangleright g) \sqcap (f \blacktriangleright g) = (\mathbf{1} \sqcup f) \blacktriangleright g = \mathbf{1} \blacktriangleright g = g. \quad \square$$

Clearly, by duality we have

- (1) $\mathbf{0} \blacktriangleleft f = f$.
- (2) $f \blacktriangleleft (g \blacktriangleleft h) = g \blacktriangleleft (f \blacktriangleleft h)$.
- (3) $f \blacktriangleleft g = g^* \blacktriangleleft f^*$.
- (4) $f \blacktriangleleft g \sqsubseteq g$ where $f, g \in \mathbf{F}_{CN}$.

To sum up the above results, the operation \blacktriangleright is a type-2 fuzzy implication on the lattice of convex normal fuzzy truth values \mathbf{F}_{CN} , and possesses similar properties that a type-1 fuzzy S-implication has.

Since interval fuzzy truth values are normal convex, and type-2 t-norms, t-conorms and negations are closed on this set, the following result is immediate. (See also [3,1].)

Corollary 17. *The operation \blacktriangleright is a type-2 fuzzy implication on the subalgebra of interval fuzzy truth values \mathbf{F}_I .*

3.1. The extended S-implications of fundamental t-norms

The three basic t-norms/t-conorms are the min/max, the product/algebraic sum and the Łukasiewicz operators. In this section we discuss the properties of extended S-implications formed by these operators. Certainly, such implications have all the previously proved properties.

First, we discuss the meet (\sqcap) and join (\sqcup), which are the extensions of \wedge and \vee . Let us introduce the notations $\blacktriangleright_{\vee}$ and $\blacktriangleleft_{\wedge}$ for the extended S-implication and coimplication formed by join and meet and an extended strong negation. Thus

$$f \blacktriangleright_{\vee} g = f^* \sqcup g = (f^* \wedge g^L) \vee ((f^*)^L \wedge g),$$

$$f \blacktriangleleft_{\wedge} g = f^* \sqcap g = (f^* \wedge g^R) \vee ((f^*)^R \wedge g).$$

Meet and join are special in the sense that they form a distributive lattice on the set of normal convex fuzzy truth values. From the absorption laws in \mathbf{F}_{CN} we have the following.

Corollary 18. For all $f, g \in \mathcal{F}_{\text{CN}}$,

$$f \blacktriangleright_{\vee} (f \blacktriangleleft_{\wedge} g) = f \blacktriangleleft_{\wedge} (f \blacktriangleright_{\vee} g) = f^*.$$

Because of the idempotency of \sqcap and \sqcup we have the following.

Proposition 19. The following distributive laws hold for all $f, g, h \in \mathcal{F}$.

- (1) $f \blacktriangleright_{\vee} (g \sqcup h) = (f \blacktriangleright_{\vee} g) \sqcup (f \blacktriangleright_{\vee} h)$; $f \blacktriangleleft_{\wedge} (g \sqcap h) = (f \blacktriangleleft_{\wedge} g) \sqcap (f \blacktriangleleft_{\wedge} h)$.
- (2) $(f \sqcap g) \blacktriangleright_{\vee} h = (f \blacktriangleright_{\vee} h) \sqcup (g \blacktriangleright_{\vee} h)$; $(f \sqcup g) \blacktriangleleft_{\wedge} h = (f \blacktriangleleft_{\wedge} h) \sqcap (g \blacktriangleleft_{\wedge} h)$.

According to [9], extended continuous and Archimedean t-norms can be calculated as follows.

Theorem 20 (Gera and Dombi [9]). If Δ is a continuous and Archimedean t-norm, then the following hold for all $f, g \in \mathcal{F}$. For all $z > 0$,

$$(f \blacktriangle g)(z) = \bigvee_{x \geq z} (f(x) \wedge g(x \triangleright z)) = \bigvee_{y \geq z} (f(y \triangleright z) \wedge g(y)). \tag{1}$$

Furthermore, for $z = 0$, if Δ is strict then

$$(f \blacktriangle g)(0) = ((f \wedge g^{\text{LR}}) \vee (f^{\text{LR}} \wedge g))(0), \tag{2}$$

and if Δ is nilpotent then

$$(f \blacktriangle g)(0) = (f \wedge (g^{\text{L}})^*)^{\text{LR}}(0), \tag{3}$$

where \triangleright denotes the residual implication of Δ , and $x' = (x \triangleright 0)$ is the strong negation corresponding to \triangleright .

We have a similar result for continuous and Archimedean t-conorms.

Theorem 21 (Gera and Dombi [9]). If ∇ is a continuous and Archimedean t-conorm, then the following hold for all $f, g \in \mathcal{F}$. For all $z < 1$,

$$(f \blacktriangledown g)(z) = \bigvee_{x \leq z} (f(x) \wedge g(x \triangleleft z)) = \bigvee_{y \leq z} (f(y \triangleleft z) \wedge g(y)). \tag{4}$$

Furthermore, for $z = 1$, if ∇ is strict then

$$(f \blacktriangledown g)(1) = ((f \wedge g^{\text{LR}}) \vee (f^{\text{LR}} \wedge g))(1), \tag{5}$$

and if ∇ is nilpotent then

$$(f \blacktriangledown g)(1) = (f \wedge (g^{\text{R}})^*)^{\text{LR}}(1), \tag{6}$$

where \triangleleft denotes the residual coimplication of ∇ , and $x' = (x \triangleleft 1)$ is the strong negation corresponding to \triangleleft .

In view of these theorems, the extended Łukasiewicz S-implication and S-coimplication are

$$\begin{aligned}
 (f \blacktriangleright_{\text{L}} g)(z) &= (f^* \blacktriangledown_{\text{L}} g)(z) = \begin{cases} \bigvee_{y \leq z} (f(1 - z + y) \wedge g(y)) & \text{if } z < 1, \\ (f \wedge g^{\text{R}})^{\text{LR}}(1) & \text{otherwise,} \end{cases} \\
 (f \blacktriangleleft_{\text{L}} g)(z) &= (f^* \blacktriangle_{\text{L}} g)(z) = \begin{cases} \bigvee_{y \geq z} (f(y - z) \wedge g(y)) & \text{if } z > 0, \\ (f \wedge g^{\text{L}})^{\text{LR}}(0) & \text{otherwise.} \end{cases}
 \end{aligned}$$

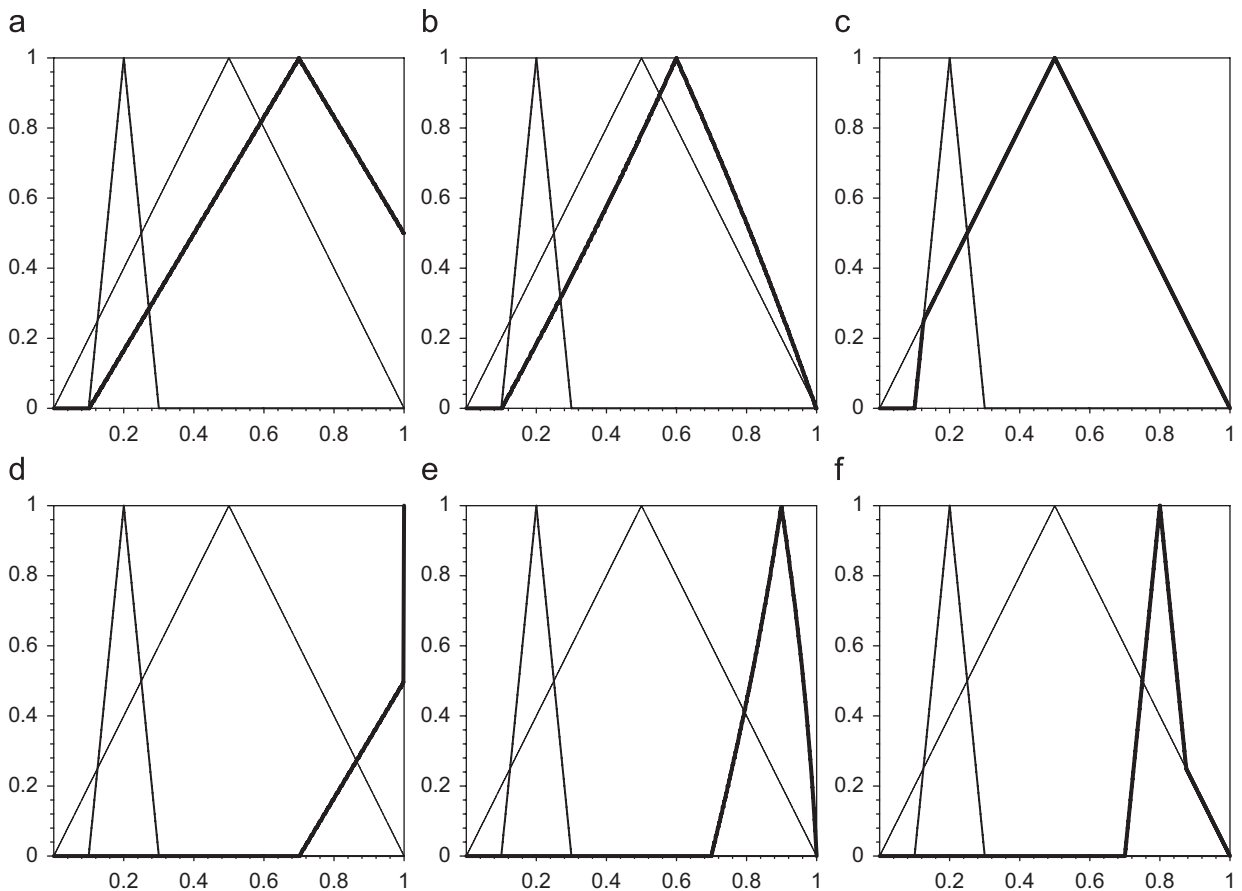


Fig. 1. Extended S-implications of the three basic t-norms on normal convex fuzzy truth values (thin line— f , normal line— g , thick line—result).

Similarly, the extended S-implication and S-coimplication of product/algebraic sum are

$$(f \blacktriangleright_P g)(z) = (f^* \blacktriangledown_P g)(z) = \begin{cases} \bigvee_{y \leq z} (f((1-z)/(1-y)) \wedge g(y)) & \text{if } z < 1, \\ ((f^* \wedge g^{LR}) \vee (f^{LR} \wedge g))(1) & \text{otherwise,} \end{cases}$$

$$(f \blacktriangleleft_P g)(z) = (f^* \blacktriangle_P g)(z) = \begin{cases} \bigvee_{y \geq z} (f((y-z)/y) \wedge g(y)) & \text{if } z > 0, \\ ((f^* \wedge g^{LR}) \vee (f^{LR} \wedge g))(0) & \text{otherwise.} \end{cases}$$

Specific calculations with these formulas can be done efficiently with a discretization of the unit interval. Fig. 1 shows the differences between these extended implications on triangular fuzzy truth values.

4. Extended residual implications and coimplications

In this section, first we introduce extensions of residual implications in general. Then, mainly because of the special role of meet and join, we examine their extended residuals in detail.

In this section let \triangleright (resp. \triangleleft) denote the residual implication (coimplication) of a t-norm Δ (resp. t-conorm ∇). Their extensions to fuzzy truth values are defined as

$$(f \blacktriangleright g)(z) = \bigvee_{z=x \triangleright y} (f(x) \wedge g(y)),$$

$$(f \blacktriangleleft g)(z) = \bigvee_{z=x \triangleleft y} (f(x) \wedge g(y)).$$

Similarly to the case of S-implications, there is a duality between residual implications and coimplications, i.e. we have

$$f \blacktriangleleft g = (f^* \blacktriangleright g^*)^*$$

for an extended strong negation $*$. Thus, the forthcoming results are proved only for implications, and can be applied to coimplications as well.

A fundamental property of residual implications is that $x \triangleright y = 1$ if and only if $x \leq y$. Due to duality, $x \triangleleft y = 0$ whenever $x \geq y$. Based on these we can establish the following.

Proposition 22. For any extended residual implication \blacktriangleright , coimplication \blacktriangleleft , and $f, g \in \mathcal{F}$,

$$(f \blacktriangleright g)(1) = (f \wedge g^R)^{LR}(1),$$

$$(f \blacktriangleleft g)(0) = (f \wedge g^L)^{LR}(0).$$

Proof. Let $f, g \in \mathcal{F}$.

$$\begin{aligned} (f \blacktriangleright g)(1) &= \bigvee_{1=x \triangleright y} (f(x) \wedge g(y)) = \bigvee_{x \leq y} (f(x) \wedge g(y)) = \bigvee_x \bigvee_{x \leq y} (f(x) \wedge g(y)) \\ &= \bigvee_x \left(f(x) \wedge \bigvee_{x \leq y} g(y) \right) = \bigvee_x (f(x) \wedge g^R(x)) = (f \wedge g^R)^{LR}(1). \end{aligned}$$

Proof is analogous for \blacktriangleleft . \square

Note that

$$\bigvee_{x \leq y} (f(x) \wedge g(y)) = \bigvee_z \bigvee_{x \leq z \leq y} (f(x) \wedge g(y)) = \bigvee_z \left(\bigvee_{x \leq z} f(x) \right) \wedge \left(\bigvee_{z \leq y} g(y) \right),$$

thus we have

$$(f \wedge g^R)^{LR} = (f^L \wedge g)^{LR} = (f^L \wedge g^R)^{LR},$$

and also by similar reasoning we have

$$(f \wedge g^L)^{LR} = (f^R \wedge g)^{LR} = (f^R \wedge g^L)^{LR}.$$

Lemma 23. The following hold for all $f \in \mathcal{F}$:

- (1) $\mathbf{1} \blacktriangleright f = f$; $\mathbf{0} \blacktriangleleft f = f$.
- (2) $f \blacktriangleright \mathbf{1} = \mathbf{0} \blacktriangleright f = f^{LR} \wedge \mathbf{1}$; $f \blacktriangleleft \mathbf{0} = \mathbf{1} \blacktriangleleft f = f^{LR} \wedge \mathbf{0}$.

Proof. Let $f \in \mathcal{F}$:

$$(\mathbf{1} \blacktriangleright f)(1) = (\mathbf{1}^L \wedge f^R)^{LR}(1) = (\mathbf{1} \wedge f^R)^{LR}(1) = f(1).$$

For all $z < 1$,

$$(\mathbf{1} \blacktriangleright f)(z) = \bigvee_{z=x \triangleright y} (\mathbf{1}(x) \wedge f(y)) = \bigvee_{z=1 \triangleright y} (\mathbf{1}(1) \wedge f(y)) = \bigvee_{z=y} f(y) = f(z).$$

Furthermore,

$$(f \blacktriangleright \mathbf{1})(1) = (f^L \wedge \mathbf{1}^R)^{LR}(1) = (f^L)^{LR}(1) = f^{LR}(1).$$

For all $z < 1$,

$$(f \blacktriangleright \mathbf{1})(z) = \bigvee_{z=x \triangleright y} (f(x) \wedge \mathbf{1}(y)).$$

According to the properties of residual implications, $z < 1$ implies $y < 1$, thus $\mathbf{1}(y) = 0$, and $(f \blacktriangleright \mathbf{1})(z) = 0$. $\mathbf{0} \blacktriangleright f$ can be proved similarly, the others follow by duality. \square

Lemma 23 imply that all extended residual implications and coimplications fulfill the necessary boundary conditions of implicative operators, i.e.

$$\begin{aligned} \mathbf{0} \blacktriangleright \mathbf{0} &= \mathbf{0} \blacktriangleright \mathbf{1} = \mathbf{1} \blacktriangleright \mathbf{1} = \mathbf{1}, & \mathbf{1} \blacktriangleright \mathbf{0} &= \mathbf{0}, \\ \mathbf{0} \blacktriangleleft \mathbf{0} &= \mathbf{1} \blacktriangleleft \mathbf{0} = \mathbf{1} \blacktriangleleft \mathbf{1} = \mathbf{0}, & \mathbf{0} \blacktriangleleft \mathbf{1} &= \mathbf{1}. \end{aligned}$$

Proposition 24. For all $f, g, h \in \mathcal{F}$,

$$f \blacktriangleright (g \blacktriangleright h) = g \blacktriangleright (f \blacktriangleright h).$$

Proof. Let $f, g, h \in \mathcal{F}$,

$$(f \blacktriangleright (g \blacktriangleright h))(z) = \bigvee_{z=x \triangleright y} \left(f(x) \wedge \bigvee_{y=u \triangleright v} (g(u) \wedge h(v)) \right) = \bigvee_{z=x \triangleright (u \triangleright v)} (f(x) \wedge g(u) \wedge h(v)).$$

A similar argument can be given for $g \blacktriangleright (f \blacktriangleright h)$, and so the statement follows from the identity $x \triangleright (u \triangleright v) = u \triangleright (x \triangleright v)$, which holds for all residual fuzzy implications. \square

Fig. 2 shows the difference between the extended residual implications of the three basic t-norms, min, product and Łukasiewicz. The formulas for the extended residual of the product and the Łukasiewicz t-norms are

$$\begin{aligned} (f \blacktriangleright_{PG})(z) &= \begin{cases} (f \wedge g^R)^{LR}(1) & \text{if } z = 1, \\ \bigvee_{x>0} (f(x) \wedge g(zx)) & \text{otherwise,} \end{cases} \\ (f \blacktriangleright_{LG})(z) &= \begin{cases} (f \wedge g^R)^{LR}(1) & \text{if } z = 1, \\ \bigvee_x (f(x) \wedge g(x + z - 1)) & \text{otherwise.} \end{cases} \end{aligned}$$

For a pointwise formula of the extended residual implication of minimum (denoted by \square) see Theorem 25 in the next subsection.

4.1. The extended residuals of \wedge and \vee

As well as the minimum (\wedge) and maximum (\vee) operators, their type-2 extensions, meet (\sqcap) and join (\sqcup) are widely used in many applications. These operations are fundamental in type-2 fuzzy logic systems, see [22] for a thorough discussion on their algebraic properties.

The residuals of \wedge and \vee have the well-known formulas

$$x \triangleright_{\wedge} y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases} \quad \text{and} \quad x \triangleleft_{\vee} y = \begin{cases} 0 & \text{if } y \leq x, \\ y & \text{otherwise.} \end{cases}$$

In this subsection we consider the extensions of \triangleright_{\wedge} and \triangleright_{\vee} . We will use the unique notation \square and \sqcup for these operators, i.e.

$$\begin{aligned} (f \square g)(z) &= \bigvee_{z=x \triangleright_{\wedge} y} (f(x) \wedge g(y)), \\ (f \sqcup g)(z) &= \bigvee_{z=x \triangleleft_{\vee} y} (f(x) \wedge g(y)). \end{aligned}$$

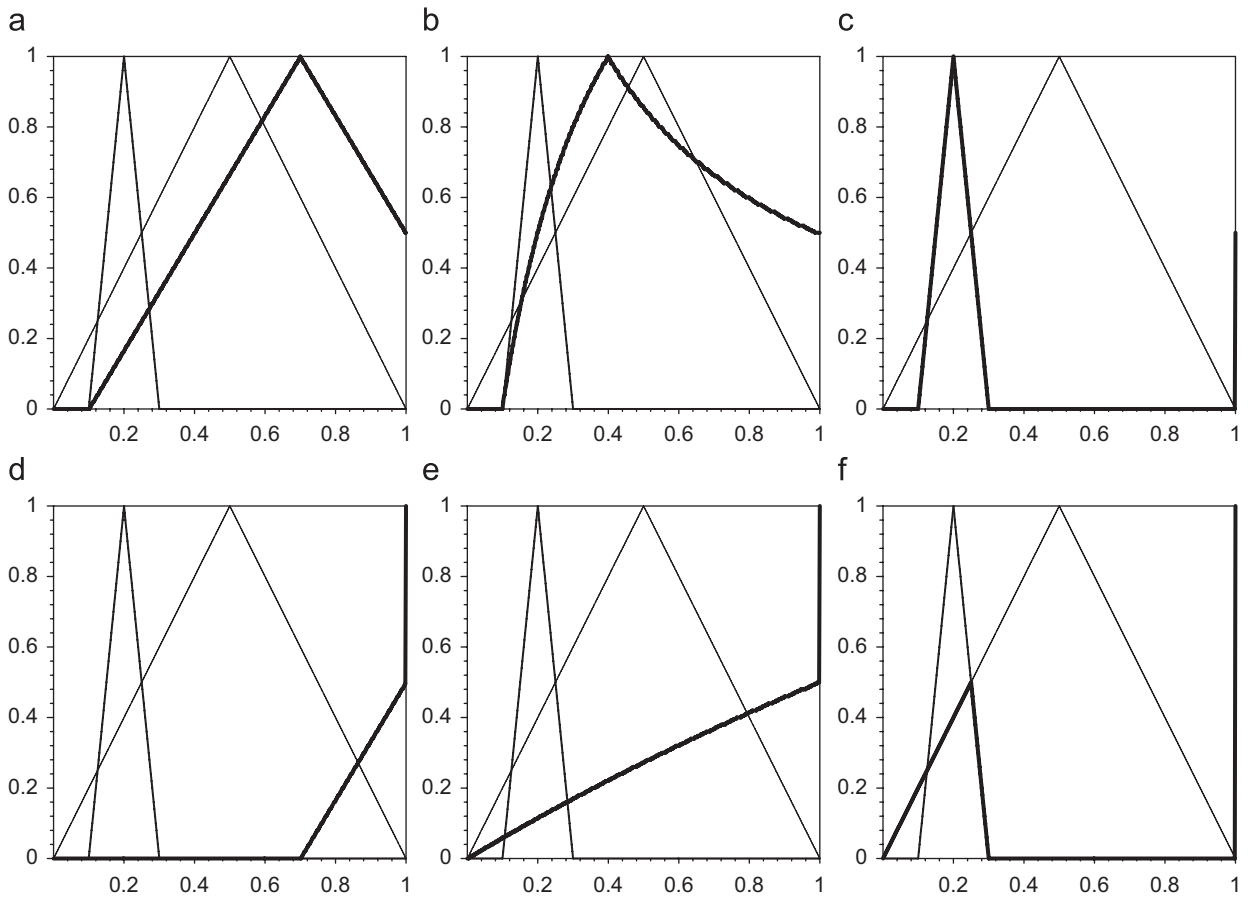


Fig. 2. Extended residual implications of the three basic t-norms on normal convex fuzzy truth values (thin line— f , normal line— g , thick line—result).

To simplify the forthcoming formulas, we introduce the following notations, which are strict counterparts of the operators R and L . For all $f \in \mathcal{F}$ let

$$f^f(x) = \begin{cases} \bigvee_{y>x} f(y) & \text{if } x < 1, \\ 0 & \text{otherwise,} \end{cases} \quad f^l(x) = \begin{cases} \bigvee_{y<x} f(y) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Note that $f^R = f^f \vee f$ and also $f^L = f^l \vee f$. As a consequence,

$$f^f \vee f^L = f^f \vee f^l \vee f = f^R \vee f^L = f^{LR}$$

and on similar considerations $f^l \vee f^R = f^{LR}$.

The operations \sqsubset and \sqsupset can be expressed in terms of pointwise operations.

This theorem will be used extensively throughout this section.

Theorem 25. For all $f, g \in \mathcal{F}$,

$$(f \sqsubset g)(z) = \begin{cases} (f \wedge g^R)^{LR}(1) & \text{if } z = 1, \\ (f^f \wedge g)(z) & \text{otherwise,} \end{cases} \quad (f \sqsupset g)(z) = \begin{cases} (f \wedge g^L)^{LR}(1) & \text{if } z = 0, \\ (f^l \wedge g)(z) & \text{otherwise.} \end{cases}$$

Proof. The case $z = 1$ follows from Proposition 22. For all $z < 1$, $(x \triangleright_{\wedge} y) = y$ and so

$$(f \sqsubset g)(z) = \bigvee_{x>z, y=z} (f(x) \wedge g(y)) = \left(\bigvee_{x>z} f(x) \right) \wedge g(z) = (f^r \wedge g)(z).$$

The formula for \sqsupset can be proved similarly. \square

It is known that $x \triangleright y = 1$ iff $x \leq y$ holds for any residual fuzzy implication. The extended counterpart of this equivalence is

$$f \blacktriangleright g = \mathbf{1} \quad \text{iff} \quad f \leq g$$

for a binary relation \leq over \mathcal{F} . Here, we give necessary and sufficient conditions in the special case of \sqsubset .

Theorem 26. For all $f, g \in \mathcal{F}$, $f \sqsubset g = \mathbf{1}$ if and only if

- (1) $f, g \in \mathcal{F}_N$, and
- (2) $g^l(x_0) = 0$, where $x_0 = \sup\{x \mid f(x) > 0\}$.

Proof. Suppose $f \sqsubset g = \mathbf{1}$. On the one hand it implies $(f \sqsubset g)(1) = 1$, i.e. by definition $(f \wedge g^R)^{LR}(1) = 1$ or equivalently $f \wedge g^R \in \mathcal{F}_N$. It is easy to see that the normality of f and g^R (and thus the normality of g) is necessary.

On the other hand, for all $z < 1$, $(f \sqsubset g)(z) = 0$ if and only if $(f^r \wedge g)(z) = 0$, by definition. Let

$$x_0 = \sup\{x \mid f(x) > 0\},$$

i.e. the least upper bound of the (not necessarily convex) support of f . It always exists, since as we have seen before, f is necessarily normal. Note that $f(x_0)$ is zero if f is right-continuous at x_0 , and non-zero otherwise. Also note that $f^r(x) = 0$ for all $x \geq x_0$. There are three cases.

- if $x_0 = 1$, i.e. $f^r(x) > 0$ for all $x < 1$, then it is necessary that $g(x) = 0$ for all $x < 1$, i.e. $g^l(1) = 0$. Note that because of normality this means that $g = \mathbf{1}$,
- if $x_0 = 0$, i.e. $f = \mathbf{0}$ (since f must be normal), then there is no constraint on the value of $g(x)$ for $x < 1$ (due to the definition of l we can write $g^l(0) = 0$),
- if $x_0 \in (0, 1)$, then it is necessary that $g(x) = 0$ for all $x < x_0$, i.e. $g^l(x_0) = 0$.

Summarized, $(f \sqsubset g)(z) = 0$ for all $z < 1$ implies $g^l(x_0) = 0$.

Now, suppose $f, g \in \mathcal{F}_N$, and $g^l(x_0) = 0$. For all $z < 1$, it is clear that $(f^r \wedge g)(z) = 0$, by considering the following cases:

- for all $z < x_0$, $g^l(x_0) = 0$ implies $g(z) = 0$,
- for all $z \geq x_0$, $f^r(z) = 0$, since x_0 is the least upper bound of the support of f .

In case $z = 1$, by definition $(f \sqsubset g)(z) = 1$ if and only if $f \wedge g^R$ is normal. By the definition of x_0 and the assumptions $g^l(x_0) = 0$ and $g \in \mathcal{F}_N$ follows that $g^R(x_0) = g^R(0) = 1$, i.e. g is constantly 1 on the interval $[0, x_0]$. Note that since f is normal and x_0 is the least upper bound of its support, the restriction of f on $[0, x_0]$ is also normal. Now, this implies that $f \wedge g^R$ is normal on $[0, x_0]$, and thus on the real unit interval, too. \square

4.2. Distributive properties of \sqsubset and \sqsupset

Proposition 27. The following distributive laws hold for all $f, g, h \in \mathcal{F}$,

- (1) $f \sqsubset (g \vee h) = (f \sqsubset g) \vee (f \sqsubset h)$; $f \sqsupset (g \vee h) = (f \sqsupset g) \vee (f \sqsupset h)$.
- (2) $(f \vee g) \sqsubset h = (f \sqsubset h) \vee (g \sqsubset h)$; $(f \vee g) \sqsupset h = (f \sqsupset h) \vee (g \sqsupset h)$.

Proof. We prove only the first equality of item 1. Item 2 can be shown analogously, and the formulas with \sqsupset follow from duality. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned} (f \sqsubset (g \vee h))(1) &= (f \wedge (g \vee h)^R)^{LR} = (f \wedge (g^R \vee h^R))^{LR} \\ &= ((f \wedge g^R) \vee (f \wedge h^R))^{LR} = (f \wedge g^R)^{LR} \vee (f \wedge h^R)^{LR} = (f \sqsubset g)(1) \vee (f \sqsubset h)(1). \end{aligned}$$

For any $x < 1$,

$$(f \sqsubset (g \vee h))(x) = (f^r \wedge (g \vee h))(x) = ((f^r \wedge g) \vee (f^r \wedge h))(x) = ((f \sqsubset g) \vee (f \sqsubset h))(x). \quad \square$$

The operations \sqsubset and \sqsupset do not distribute over \wedge in general, only the following inequalities hold.

Proposition 28. For all $f, g, h \in \mathcal{F}$,

- (1) $f \sqsubset (g \wedge h) \leq (f \sqsubset g) \wedge (f \sqsubset h)$; $f \sqsupset (g \wedge h) \leq (f \sqsupset g) \wedge (f \sqsupset h)$.
- (2) $(f \wedge g) \sqsubset h \leq (f \sqsubset h) \wedge (g \sqsubset h)$; $(f \wedge g) \sqsupset h \leq (f \sqsupset h) \wedge (g \sqsupset h)$.

Proof. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned} (f \sqsubset (g \wedge h))(1) &= (f^L \wedge g \wedge h)^{LR} = ((f^L \wedge g) \wedge (f^L \wedge h))^{LR} \\ &\leq (f^L \wedge g)^{LR} \wedge (f^L \wedge h)^{LR} = (f \sqsubset g)(1) \wedge (f \sqsubset h)(1). \end{aligned}$$

For any $x < 1$,

$$(f \sqsubset (g \wedge h))(x) = (f^r \wedge g \wedge h)(x) = (f^r \wedge g \wedge f^r \wedge h)(x) = (f \sqsubset g)(x) \wedge (f \sqsubset h)(x).$$

Item 2 can be proved similarly, and the formulas with \sqsupset follow by duality. \square

In general, \sqsubset does not distribute over \sqcap and \sqcup , only the following inequalities hold.

Theorem 29. For all $f, g, h \in \mathcal{F}$,

$$f \sqsubset (g \sqcap h) \leq (f \sqsubset g) \sqcap (f \sqsubset h); \quad f \sqsubset (g \sqcup h) \leq (f \sqsubset g) \sqcup (f \sqsubset h).$$

To prove this theorem we need the following two lemmas.

Lemma 30. For all $f, g \in \mathcal{F}$,

$$(f \sqsubset g)^R = (f^R \wedge g)^R \vee (f \wedge g^R)^{LR}, \tag{8}$$

$$(f \sqsubset g)^L(z) = \begin{cases} (f^{LR} \wedge g^{LR})(1) & \text{if } z = 1, \\ (f^r \wedge g)^L(z) & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Let $f, g \in \mathcal{F}$.

$$(f \sqsubset g)^R(1) = (f \sqsubset g)(1) = (f \wedge g^R)^{LR}(1).$$

For all $x < 1$,

$$(f \sqsubset g)^R(x) = \bigvee_{y \geq x} (f \sqsubset g)(y) = (f \sqsubset g)(1) \vee \bigvee_{1 > y \geq x} (f \sqsubset g)(y).$$

But since $(f^r \wedge g)(1) = 0$,

$$\bigvee_{1 > y \geq x} (f \sqsubset g)(y) = \bigvee_{y \geq x} (f^r \wedge g)(y) = (f^r \wedge g)^R(x),$$

thus

$$(f \sqsubset g)^R = (f^r \wedge g)^R \vee (f \wedge g^R)^{LR}.$$

Since $(f \wedge g)^R \leq (f \wedge g^R)^{LR}$, we have

$$\begin{aligned} (f^r \wedge g)^R \vee (f \wedge g^R)^{LR} &= (f^r \wedge g)^R \vee (f \wedge g)^R \vee (f \wedge g^R)^{LR} = ((f^r \wedge g) \vee (f \wedge g))^R \vee (f \wedge g^R)^{LR} \\ &= ((f^r \vee f) \wedge g)^R \vee (f \wedge g^R)^{LR} = (f^R \wedge g)^R \vee (f \wedge g^R)^{LR}. \end{aligned}$$

In case $x < 1$, the formula for $(f \sqsubset g)^L(x)$ is straightforward from the definition of \sqsubset . Also by definition,

$$(f \sqsubset g)^L(1) = (f \sqsubset g)^{LR}(1) = ((f^r \wedge g)^{LR} \vee (f^L \wedge g)^{LR})(1).$$

Furthermore,

$$\begin{aligned} (f^r \wedge g)^{LR} \vee (f^L \wedge g)^{LR} &= ((f^r \wedge g) \vee (f^L \wedge g))^{LR} = ((f^r \vee f^L) \wedge g)^{LR} \\ &= (f^{LR} \wedge g)^{LR} = f^{LR} \wedge g^{LR}. \quad \square \end{aligned}$$

Lemma 31. For all $f, g \in \mathcal{F}$,

$$f \sqsubset g \leq f \sqsubset g^R \leq (f \sqsubset g)^R, \tag{10}$$

$$f \sqsubset g \leq f \sqsubset g^L \leq (f \sqsubset g)^L. \tag{11}$$

Proof. Recall Theorem 25, the pointwise formulas for \sqsubset . The first inequality of (10) holds, because $(f \sqsubset g)(1) = (f \sqsubset g^R)(1)$ by definition, and $f^r \wedge g \leq f^r \wedge g^R$. As for the second inequality, $(f \sqsubset g^R)(1) = (f \sqsubset g)^R(1)$ also by definition. For all $x < 1$, the inequality

$$(f^r \wedge g^R)(x) \leq ((f^R \wedge g)^R \vee (f \wedge g^R)^{LR})(x)$$

holds, as shown by the following reasoning:

$$\begin{aligned} f^r \wedge g^R &\leq f^R \wedge g^R = f^R \sqcap g^R = (f \sqcap g)^R = ((f^R \wedge g) \vee (f \wedge g^R))^R \\ &= (f^R \wedge g)^R \vee (f \wedge g^R)^R \leq (f^R \wedge g)^R \vee (f \wedge g^R)^{LR}. \end{aligned}$$

The first inequality of (11) can be proved analogously as above. Furthermore,

$$(f \sqsubset g^L)(1) = (f \sqsubset g)^L(1),$$

since $(f \wedge g^{LR})^{LR} = f^{LR} \wedge g^{LR}$. So, to prove the second inequality of (11) we need to show that

$$f^r \wedge g^L \leq (f^r \wedge g)^L. \tag{12}$$

Suppose (12) does not hold, i.e.

$$\exists x : (f^r \wedge g)^L(x) < (f^r \wedge g^L)(x) \quad \text{i.e.}$$

$$\exists x : (f^r \wedge g)^L(x) < f^r(x) \quad \text{and} \quad (f^r \wedge g)^L(x) < g^L(x). \tag{13}$$

By the definition of the operator L , the first inequality of (13) is equivalent to

$$\forall y \leq x : f^r(y) < f^r(x) \quad \text{or} \quad g(y) < f^r(x).$$

Since $f^r \in \mathcal{F}^-$, $f^r(y) < f^r(x)$ cannot hold for any $y \leq x$, and so $g(y) < f^r(x)$ must hold for all $y \leq x$, i.e. the first inequality of (13) implies

$$g^L(x) < f^r(x). \tag{14}$$

Analogously, the second inequality of (13) is equivalent to

$$\forall y \leq x : f^r(y) < g^L(x) \quad \text{or} \quad g(y) < g^L(x).$$

Now, since $g(y) < g^L(x)$ cannot hold for all $y \leq x$, it implies that

$$\exists y \leq x : f^r(y) < g^L(x). \tag{15}$$

So, by combining (14) and (15), (13) implies

$$\exists x : g^L(x) < f^r(x) \quad \text{and} \quad \exists y \leq x : f^r(y) < g^L(x).$$

This is a contradiction, because $f^r \in \mathcal{F}^-$, i.e. (12) and so (11) holds. \square

Now we can prove Theorem 29.

Proof. Let $f, g, h \in \mathcal{F}$.

$$\begin{aligned} f \sqsubset (g \sqcap h) &= f \sqsubset ((g \wedge h^R) \vee (g^R \wedge h)) = (f \sqsubset (g \wedge h^R)) \vee (f \sqsubset (g^R \wedge h)) \\ &\leq ((f \sqsubset g) \wedge (f \sqsubset h^R)) \vee ((f \sqsubset g^R) \wedge (f \sqsubset h)) \leq ((f \sqsubset g) \wedge (f \sqsubset h)^R) \vee ((f \sqsubset g)^R \wedge (f \sqsubset h)) \\ &= (f \sqsubset g) \sqcap (f \sqsubset h). \end{aligned}$$

The other equality follows analogously. \square

In the next subsections we investigate the operation \sqsubset on the main subalgebras of \mathbf{F} .

4.3. Convex and normal fuzzy truth values

The most elementary subset of \mathcal{F} is the set of singleton fuzzy truth values \mathcal{F}_S . A fuzzy truth value f_x is a singleton if there exists exactly one $x \in \mathcal{I}$ such that $f_x(x) = 1$, and for all $y \neq x$, $f_x(y) = 0$. It is proved that $(\mathcal{F}_S, \sqcap, \sqcup, *, \mathbf{0}, \mathbf{1})$ is isomorphic to the algebra of (type-1) truth values $(\mathcal{I}, \wedge, \vee, ', 0, 1)$ by the bijection $x \mapsto f_x$ from \mathcal{I} to \mathcal{F}_S .

It is easy to see that the elements of \mathcal{F}_S are normal convex, it is closed w.r.t \sqcap, \sqcup , and the partial orders \sqsubseteq and \preceq coincide (in fact, $(\mathcal{F}_S, \sqsubseteq)$ is a chain).

Proposition 32. For all $f_x, g_y \in \mathcal{F}_S$,

$$f_x \sqsubseteq g_y \quad \text{if and only if} \quad x \leq y, \tag{16}$$

$$f_x \sqsubset g_y = \begin{cases} \mathbf{1} & \text{if } f_x \sqsubseteq g_y, \\ g_y & \text{otherwise.} \end{cases} \tag{17}$$

Proof is straightforward from the formulas of \sqsubseteq and \sqsubset . Having $f_x \sqsubset g_y$, the following theorem can be established stating that the algebras \mathbf{I} and \mathbf{F} equipped with the residual implication \triangleright_{\wedge} and its extension are also isomorphic.

Theorem 33. The algebra $\mathbf{F}_S = (\mathcal{F}_S, \sqcap, \sqcup, *, \mathbf{0}, \mathbf{1})$ is isomorph to $\mathbf{I} = (\mathcal{I}, \wedge, \triangleright_{\wedge}, ', 0, 1)$, where \triangleright_{\wedge} denotes the residual implication of \wedge .

It is easy to see that \sqsubset is a type-2 fuzzy implication on \mathbf{F}_S , i.e. besides the boundary conditions, it is antitone/monotone. However, it is an open question whether it is the largest such subalgebra of \mathbf{F} containing \mathbf{F}_S .

An undoubtedly important and recently most popular subalgebra of \mathbf{F} is the algebra of interval fuzzy truth values \mathbf{F}_I . It is proved to be isomorphic to the algebra $(I^{[2]}, \wedge, \vee, ', 0, 1)$, where $I^{[2]}$ denotes the set of closed intervals in I . Thus, the elements of \mathbf{F}_I can be represented by a pair $(a, b) \in I^{[2]}$. It is important to remark that $I^{[2]}$ contains only closed intervals, and so the elements of \mathbf{F}_I are closed interval fuzzy truth values. In fact, the proof of the next theorem is based on this observation.

Theorem 34. The algebra \mathbf{F}_I of interval fuzzy truth values is not closed w.r.t. \sqsubset and \sqsupset .

Proof. We prove by example. Let $f, g \in \mathbf{F}_I$ be represented by the intervals $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{1}{2}, \frac{3}{4})$, respectively. Then $f \sqsubset g$ is not a closed interval. Moreover, it is not even an interval fuzzy truth value. Indeed, for all $x < 1$, $(f \sqsubset g)(x) =$

$(f^r \wedge g)(x)$, and $(f \sqsubset g)(\frac{2}{3}) = f^r(\frac{2}{3}) \wedge g(\frac{2}{3}) = 0$ while for all $\frac{1}{2} \leq y < \frac{2}{3}$, $(f \sqsubset g)(y) = 1$. To see that it is not even an interval, note that $(f \sqsubset g)(1) = 1$. A similar example can be given for \sqsupset . \square

By this negative result on \mathbf{F}_I it is natural to ask the following. What is the largest subalgebra \mathbf{A} of \mathbf{F} containing \mathbf{F}_S such that for all $f, g \in \mathbf{A}$, $f \sqsubset g$ reduces to

$$f \sqsubset g = \begin{cases} \mathbf{1} & \text{if } f \sqsubseteq g, \\ g & \text{otherwise.} \end{cases}$$

Since interval fuzzy truth values are convex and normal, it is straightforward to investigate the latter. In fact, the following corollary is immediate from the proof of theorem 34 since the result there is not even convex.

Corollary 35. *The set \mathcal{F}_C of convex fuzzy truth values is not closed w.r.t. \sqsubset and \sqsupset .*

We have a positive result on normal fuzzy truth values.

Theorem 36. *The set \mathcal{F}_N of normal fuzzy truth values is closed w.r.t. \sqsubset and \sqsupset . Moreover, $f \sqsubset g \in \mathcal{F}_N$ (resp. $f \sqsupset g \in \mathcal{F}_N$) if and only if $f, g \in \mathcal{F}_N$.*

Proof. By Lemma 30 we have

$$(f \sqsubset g)^L(1) = (f^{LR} \wedge g^{LR})(1) = f^{LR}(1) \wedge g^{LR}(1) = f^L(1) \wedge g^L(1),$$

so $(f \sqsubset g)^L(1) = 1$ if and only if $f^L(1) = 1$ and $g^L(1) = 1$. The operation \sqsubset preserves normality by duality. \square

It is proved in [22] that the lattice of normal convex fuzzy truth values is a maximal lattice in \mathbf{F} . Since the operation \sqsubset is not closed on \mathcal{F}_C , the next theorem is straightforward.

Theorem 37. *The operations \sqsubset and \sqsupset do not form an adjoint pair on the lattice $(F_{CN}, \sqcap, \sqcup, \sqsubset, \mathbf{0}, \mathbf{1})$.*

Naturally arises the following. What is the largest sublattice of F_{CN} for which \sqsubset and \sqsupset are an adjoint pair? The question is correct, since for example, on the algebra \mathbf{F}_S of singleton fuzzy truth values \sqsubset is the residual of \sqsupset .

4.4. Left- and right-maximal fuzzy truth values

In the following let \mathcal{F}_{LM} and \mathcal{F}_{RM} denote the sets of left-maximal and right-maximal fuzzy truth values.

Proposition 38. *If $f \in \mathcal{F}_{RM}$, then*

$$(f \sqsubset g)(x) = \begin{cases} (f \wedge g^R)^{LR}(1) & \text{if } x = 1, \\ (f^{LR} \wedge g)(x) & \text{otherwise.} \end{cases} \quad (18)$$

Furthermore, if also $g \in \mathcal{F}_{RM}$, then it simplifies to

$$f \sqsubset g = f^{LR} \wedge g, \quad (19)$$

and so $f \sqsubset g \in \mathcal{F}_{RM}$. If $g \in \mathcal{F}_{LM}$, then $f \sqsubset g \in \mathcal{F}_{LM}$.

Proof. The right-maximality of f implies $f^r(x) = f^{LR}$ for all $x < 1$, hence the formula $(f^{LR} \wedge g)(x)$ in case $x < 1$. If in addition g is also right-maximal, then

$$(f \wedge g^R)^{LR}(1) = (f \wedge g^{LR})^{LR}(1) = (f^{LR} \wedge g^{LR})(1) = (f^{LR} \wedge g)(1).$$

It is easy to see that in this case $f^{LR} \wedge g$ is right-maximal, too.

The left-maximality of $f \sqsubset g$ assuming g is left-maximal is as follows. According to (18), $f \sqsubset g$ is left-maximal if $(f \sqsubset g)(1) \leq (f \sqsubset g)(0)$, i.e.

$$(f \wedge g^R)^{LR} \leq (f^{LR} \wedge g)(0) = f^{LR} \wedge g^{LR}.$$

It holds for all f, g , since $(f \wedge g)^{LR} \leq f^{LR} \wedge g^{LR}$. \square

By (19), it is easy to check the following.

Corollary 39. *If $f, g \in \mathcal{F}_+$, then $f \sqsubset g \in \mathcal{F}_+$, i.e. \sqsubset is closed on \mathcal{F}_+ .*

Proposition 40. *If $f \in \mathcal{F}_{LM}$, then*

$$(f \sqsubset g)(z) = \begin{cases} f^{LR} \wedge g^{LR} & \text{if } z = 1, \\ (f^r \wedge g)(z) & \text{otherwise,} \end{cases} \tag{20}$$

moreover, it is right-maximal for all $g \in \mathcal{F}$.

Proof. Consider the following inequalities. For all $x < 1$,

$$(f^r \wedge g)(x) \leq (f^R \wedge g^R)(x) \leq (f^R \wedge g^R)(0) = f^R(0) \wedge g^R(0) = (f \sqsubset g)(1). \quad \square$$

Summarizing the above results, the following theorem holds.

Theorem 41. *The algebra $\mathbf{F}_M = (\mathcal{F}_{LM} \cup \mathcal{F}_{RM}, \sqcap, \sqcup, *, \sqsubset, \mathbf{0}, \mathbf{1})$ of left- or right-maximal fuzzy truth values is a subalgebra of $(\mathcal{F}, \sqcap, \sqcup, *, \sqsubset, \mathbf{0}, \mathbf{1})$.*

Proof. According to the previous propositions, the operation \sqsubset is closed on the union set of left- or right-maximal functions. $\mathbf{0}$ and $\mathbf{1}$ are clearly elements of it, and it is also easy to check that \sqcap and \sqcup are also closed on \mathbf{F}_M . \square

5. Conclusions

Current research is just starting to discover the rich structure of fuzzy truth values. In this paper we have discussed extended fuzzy implications from two distinct views: extended fuzzy S-implications and S-coimplications and the extended residual implications and coimplications (especially the extended residual of \wedge).

First, we have discussed type-2 S-implication operators which have a well established background due to recent literature on type-2 t-norms and t-conorms, in particular to meet and join. Our investigations have shown that type-2 S-implications have similar properties on the lattice of convex normal fuzzy truth values like type-1 S-implications on the unit interval.

Residual implications (i.e. residuals of conjunctive operations) are fundamental in fuzzy logic. A (type-1) residual implication can be extended to the set of fuzzy truth values. In Section 4 we have discussed such extended operations, in particular the operation \sqsubset , the extended residual of \wedge .

The following questions are left open for further research:

- (1) What is the adjoint operation of \sqsubset (the extended residual implication of \wedge) on \mathbf{F} ?
- (2) What is the largest subalgebra of \mathbf{F} , where \sqsubset is a type-2 fuzzy implication (Definition 10)?
- (3) What is the largest subalgebra of \mathbf{F} , where \sqsubset and the residual of \sqcap coincide?

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