Towards a General Class of Operators for Fuzzy Systems

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***Abstract*— Our starting point is the multiplicative utility function which is extensively used in the theory of multicriteria decision making. Its associativity is shown and as its generalization a class of operators is introduced with fine and useful properties. As special cases it reduces to well-known operators of fuzzy set theory: min/max, product, Einstein, Hamacher, Dombi and drastic. As a consequence, we generalize the addition of velocities in Einstein's special relativity theory to multiple moving objects. Also, a new form of the Hamacher operator is given, which makes multi-argument calculations easier. We examine the De Morgan identity which connects the conjunctive and disjunctive operators by a negation. It is shown that in some special cases (min/max, drastic and Dombi) the operator class forms a De Morgan triple with any involutive negation.**

***Index Terms*— fuzzy operators, hedges, membership function**

I. INTRODUCTION

IN the last decades many operators were introduced. The most important ones can be found in many handbooks or monographs dealing with fuzzy logic. The Dombi operator is a special one because the sign of its parameter determines the type of the operator i.e. whether it is conjunctive or disjunctive. From the practical point of view there is a great interest in such parametrical families. There are two main reasons:

- In applications, by changing a single parameter a different logic can be modeled.
- Through a learning algorithm the appropriate parameter and therefore the appropriate operator can be found.

In his first paper on fuzzy sets Zadeh [1] suggested to use the minimum and the product. Hamacher [2] discovered that operators can be generated using the solution of the associativity functional equations. Based on the result of Kuwagaki [3], Hamacher got the rational form of conjunctive and disjunctive operators. By that time, researchers working on fuzzy logic discovered that they can use a more general framework, i.e. triangular norms. The history of triangular norms started with the paper "Statistical Metric Spaces" by Menger [4]. The terms "t-norm" and "t-conorm" were introduced by Schweizer and Sklar [5]. Their basic theory has rather independent roots, namely the theory of functional equations and the theory of groups and semigroups. For a recent reference on triangular norms we recommend Klement et al. [6], more details on further generalizations in particular aggregation operators can be found in Calvo et al. [7]. Despite the wide range of operators in fact only a few are actually used. The main classes of the operators are:

1) min-max, which is widely used in fuzzy control,
2) drastic operators, which is the closest one to binary logic,
3) the strict operators, which play an important role in engineering applications. The main strict operators are the product, Hamacher, Frank, Einstein, and Dombi operators.
4) the nilpotent operators.

In this article we concentrate on the first three classes of operators. Our main objective is to introduce a class of generalized operators which includes most well-known operators. The class contains a two-parametrical family of operators which generalize the Dombi operators by preserving its main properties. This operator class contains the Hamacher and Einstein operators, and as a limit we can get the min-max and drastic operators, too.

As a corollary of the multivariate Einstein operator we get the closed form of the additivity law of velocities in the framework of special relativity theory. We give a new form of the Hamacher operator family, with which its multivariate case can be handled more easily. The De Morgan law is also studied and we give a necessary and sufficient condition for it.

II. PRELIMINARIES

T-norms are commutative, associative and monotone operations on the real unit interval with 1 as unit element. T-conorms are in some sense dual to t-norms. A t-conorm is a commutative, associative and monotone operation with 0 as unit element. In this paper, besides the min/max and the drastic operators, we are concerned with strict t-norms and t-conorms.

Because the strict t-norm and t-conorm are special cases of the general t-norm and t-conorm classes we refer to them later in this article as *conjuctive* and *disjunctive operators* denoted by $c(x, y)$, $d(x, y)$ respectively.

There are other reasons also to use this terminology:

1) We do not use pseudo inverse and ordinal sum to construct a general t-norm and t-conorm.
2) We do not use the commutativity axiom of t-norm and t-conorm.

3) We do not use the boundary condition of t-norm and t-conorm only the compatibility with the binary logic.

Definition 1: A t-norm $c : [0,1]^2 \to [0,1]$ is *strict* if and only if it is continuous, and there exists a continuous and strictly decreasing function $f_c : [0,1] \to [0,\infty]$ with $f_c(1) = 0$ and $\lim_{x\to 0} f_c(x) = \infty$ s.t.

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right).$$

This representation is based on the following theorem of Aczél [8].

Theorem 2: A continuous and strictly monotonic function $F : [a,b]^2 \to [a,b]$ is associative if and only if

$$F(x,y) = f^{-1}\left(f(x) + f(y)\right), \tag{1}$$

where $f : [a,b] \to [0,\infty)$ is strictly decreasing. Here f is called a *generator function* of F, and F is uniquely determined up to constant multiplier of f.

Analogously, a t-conorm d is strict if and only if it is continuous and there exists a continuous and strictly increasing function $f_d : [0,1] \to [0,\infty]$ with $f_d(0) = 0$ and $\lim_{x\to 1} f_d(x) = \infty$ s.t. $d(x,y) = f_d^{-1}\left(f_d(x) + f_d(y)\right)$. We distinguish between conjunctive operators (resp. strict t-norms) and disjunctive operators (resp. strict t-conorms).

Strong negations are order reversing automorphisms of the unit interval. The usual requirements for a negation n are the following.

N1: $n : [0,1] \to [0,1]$ is continuous,
N2: n is strictly decreasing,
N3: $n(0) = 1$ and $n(1) = 0$,
N4: $n(x)$ is involutive i.e. $n(n(x)) = x$.

Because in this article we deal only with strong negations we refer to them as *negation*.

A negation n and a conjunctive operator c and a disjunctive operator d fulfill the De Morgan law if

$$d(x,y) = n(c(n(x), n(y))).$$

III. THE MULTIPLICATIVE UTILITY FUNCTION

In their seminal treatment of multiattribute utility theory (MAU), Keeney and Raiffa [9] show how certain conditions of independence among attributes yield the so called *multiplicative multiattribute utility function*

$$u_M(\mathbf{z}) = \frac{1}{k}\left(\prod_{i=1}^{n}\left(1 + k k_i u_i(z_i)\right) - 1\right) \tag{2}$$

where $\mathbf{z} = (z_1, \ldots, z_n)$, $u_i : \mathbb{R} \to [0,1]$ are utility functions, z_i are evaluations, k_i is the weight of the ith criterion, and k is a scaling constant. The formula can also be expanded as

$$\begin{aligned} u_M(\mathbf{z}) = \sum_{i=1}^{n} k_i u_i(z_i) + k \sum_{i<j} k_i k_j u_i(z_i) u_j(z_j) + \\ + k^2 \sum k_i k_j k_l u_i(z_i) u_j(z_j) u_l(z_l) + \ldots \\ + k^{n-1} k_1 k_2 \ldots k_n u_1(z_1) \ldots u_n(z_n). \end{aligned} \tag{3}$$

allowing also for $k = 0$.

Lemma 3: If $k = 0$ then

$$u_M(\mathbf{z}) = \sum_{i=1}^{n} k_i u_i(z_i). \tag{4}$$

Proof. By substituting $k = 0$ into the expanded formula of u_M (3) we get the result. ■

The utility function is *normal* if $u_M(\mathbf{z}) = 0$ whenever $u_i(z_i) = 0$, and $u_M(\mathbf{z}) = 1$ whenever $u_i(z_i) = 1$ for all $i \in \{1, \ldots, n\}$. The normality of the utility functions implies

$$1 + k = \prod_{i=1}^{n}(1 + k k_i), \tag{5}$$

i.e. assuming the normality of u_M, k is determined only by the weights k_i.

A. The Associativity of the Multiplicative Utility Function

Let us substitute $x_i := k_i u_i(z_i)$ in Eq. (2). Then the transformed multiplicative utility function is

$$u_M^*(\mathbf{x}) = \frac{1}{k}\left(\prod_{i=1}^{n}(1 + k x_i) - 1\right). \tag{6}$$

Theorem 4: The function u_M^* is associative.

Proof. The proof is based on the representation theorem of Aczél. It can be easily verified, that (6) can also be written in the form (1), by putting

$$f(x) = \ln(1 + kx), \tag{7}$$

and

$$f^{-1}(x) = \frac{1}{k}\left(e^x - 1\right), \tag{8}$$

moreover, f fulfills the requirements of Theorem 2. ■

B. Logical operators and the Multiplicative Utility Function

Let $g : [0,1] \to [0,\infty]$ be a generator function of a strict operator. Let

$$f(x) = \ln(1 + \gamma g(x)), \tag{9}$$

and so

$$f^{-1}(x) = g^{-1}\left(\frac{1}{\gamma} e^x - 1\right). \tag{10}$$

Note, that for all $\gamma \in (0,\infty)$, f fulfills the requirements of a generator function of a strict operator (see for example [10]). By Aczél's theorem, the associative operator $o : [0,1]^n \to [0,1]$ generated by f is

$$o(x_1, \ldots, x_n) = g^{-1}\left(\frac{1}{\gamma}\left(\prod\left(1 + \gamma g(x_i)\right) - 1\right)\right). \tag{11}$$

Similarly to (3), by first expanding the argument of g^{-1} to

$$\begin{aligned} \sum_{i=1}^{n} g(x_i) + \gamma \sum_{i<j} g(x_i) g(x_j) + \\ + \gamma^2 \sum g(x_i) g(x_j) g(x_l) + \ldots \\ + \gamma^{n-1} g(x_1) \ldots g(x_n), \end{aligned}$$

we can put

$$o(x_1,\ldots,x_n)|_{\gamma=0} = g^{-1}\left(\sum g(x_i)\right), \tag{12}$$

thus the case $\gamma = 0$ also results in a strict operator. Next, we will show that different types of operators fit into the framework depending on the choice of function f. From now on, let us assume that

$$g(x) = \left(\frac{1-x}{x}\right)^{\alpha},$$

i.e., that g is the generator function of the Dombi operator with parameter α.

IV. THE GENERALIZED DOMBI OPERATOR

Definition 5: The generator functions of the Generalized Dombi operator are

$$f_c(x) = \ln\left(1+\gamma_c\left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha > 0 \tag{13}$$

$$f_d(x) = \ln\left(1+\gamma_d\left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha < 0 \tag{14}$$

where $\gamma_c, \gamma_d \in [0,\infty]$. From

$$c(\mathbf{x}) = f_c^{-1}\left(\sum_{i=1}^{n} f_c(x_i)\right),$$

$$d(\mathbf{x}) = f_d^{-1}\left(\sum_{i=1}^{n} f_d(x_i)\right),$$

and

$$f_c^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_c}\left(e^x-1\right)\right)^{1/\alpha}}, \qquad \alpha > 0 \tag{15}$$

$$f_d^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_d}\left(e^x-1\right)\right)^{1/\alpha}}, \qquad \alpha < 0 \tag{16}$$

the operators are given by

$$c_{GD,\gamma_c}^{(\alpha)}(\mathbf{x}) = \frac{1}{1+D_{\gamma_c}(\mathbf{x})}, \qquad \alpha > 0 \tag{17}$$

$$d_{GD,\gamma_d}^{(\alpha)}(\mathbf{x}) = \frac{1}{1+D_{\gamma_d}(\mathbf{x})}, \qquad \alpha < 0 \tag{18}$$

where $\gamma_c, \gamma_d \in [0,\infty]$ and

$$D_{\gamma}(\mathbf{x}) = \left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}. \tag{19}$$

Equations (17) and (18) differ only in the sign of α and so the Generalized Dombi operator class is:

$$o_{GD,\gamma}^{(\alpha)}(\mathbf{x}) = \frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{20}$$

In the forthcoming sections, we will show that $o_{GD,\gamma}^{(\alpha)}$ is a strict operator for $\alpha \in (-\infty,\infty)$ and $\gamma \in (0,\infty)$.

V. THE DOMBI OPERATOR CASE

The Dombi operator has the form (see [10])

$$o_D^{(\alpha)}(\mathbf{x}) = \frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \tag{21}$$

and if $\alpha > 0$ then the operator is conjunctive and if $\alpha < 0$ then the operator is disjunctive. The next Corollary follows from Lemma 3, by the substitution $k = \gamma$.

Corollary 6: The Dombi operator is a member of the Generalized Dombi operator, i.e. if $\gamma_c = \gamma_d = 0$ then

$$c_{GD,0}^{(\alpha)}(\mathbf{x}) = c_D^{(\alpha)}(\mathbf{x}), \tag{22}$$

$$d_{GD,0}^{(\alpha)}(\mathbf{x}) = d_D^{(\alpha)}(\mathbf{x}). \tag{23}$$

VI. THE HAMACHER OPERATOR CASE

Hamacher [2] was one of the first who discussed how new logical operators can be generated using the solutions of the associativity functional equation. As we have seen, with the help of the generator function of the operator, infinitely many operators can be constructed. To restrict the solution space Hamacher added a new requirement, namely he looked for operators which can be written in rational form (a quotient of two polynomials). Kuwagaki [3] showed that in this case the generator function can only have the following two forms:

$$f^{-1}(x) = \frac{ax+b}{cx+d} \quad \text{or} \quad f^{-1}(x) = \frac{ae^x+b}{ce^x+d} \tag{24}$$

Hamacher showed that to have conjunctive or disjunctive operators equations (24) have the following solutions

$$f_c^{-1}(x) = \frac{e^x}{\gamma+(1-\gamma)e^x} \tag{25}$$

$$f_d^{-1}(x) = \frac{e^x-1}{\gamma'+e^x} \tag{26}$$

The Hamacher operators are

$$c_{H,\gamma}(x,y) = \frac{xy}{\gamma+(1-\gamma)(x+y-xy)} \tag{27}$$

$$d_{H,\gamma'}(x,y) = \frac{x+y-(1-\gamma')xy}{1+\gamma' xy} \tag{28}$$

where $0 \le \gamma$ and $-1 \le \gamma'$. Let $\gamma_c = 1/\gamma$, and $\gamma_d = 1/\left(\gamma'+1\right)$.

Theorem 7: The Hamacher operator class is a special case of the Generalized Dombi operator if $\alpha = \pm 1$ and $\gamma_c, \gamma_d \in [0,\infty)$.

$$c_{GD,\gamma_c}^{(1)}(\mathbf{x}) = c_{H,\gamma}(\mathbf{x}) \tag{29}$$

$$d_{GD,\gamma_d}^{(-1)}(\mathbf{x}) = d_{H,\gamma'}(\mathbf{x}). \tag{30}$$

Proof. The inverse generator function of the Hamacher operator in the conjunctive case can be transformed into

$$f_c^{-1}(x) = \frac{e^x}{\gamma+(1-\gamma)e^x} = \frac{1}{\gamma e^{-x}+(1-\gamma)} =$$

$$= \frac{1}{1+\gamma(e^{-x}-1)}.$$

By theorem 2, $f_c^{-1}(x)$ generates the same operator as $f_c^{-1}(Ax)$ where $A \neq 0$ is a constant. Let us choose $A = -1$, so

$$f_c^{-1}(x) = \frac{1}{1 + \frac{1}{\gamma_c}(e^x - 1)} \tag{31}$$

which is the same as the inverse generator function of Generalized Dombi operator in the $\alpha = 1$ case.

The inverse generator function of the disjunctive Hamacher operator can be transformed similarly by

$$f_d^{-1}(x) = \frac{e^x - 1}{\gamma' + e^x} = \frac{e^x - 1}{e^x - 1 + \gamma' + 1} = \\ = \frac{1}{1 + (\gamma' + 1)\left(e^x - 1\right)^{-1}}.$$

Since $\gamma' + 1 = 1/\gamma_d$, so

$$f_c^{-1}(x) = \frac{1}{1 + \frac{1}{\gamma_d}\left(e^x - 1\right)^{-1}} \tag{32}$$

which is the same as the inverse generator function of the Generalized Dombi operator in the $\alpha = -1$ case.

It is easy to see that $0 < \gamma$ and $-1 < \gamma'$ are the same requirements as $\gamma_c, \gamma_d \in (0, \infty)$. ■

Corollary 8: Using the new type of generator functions we can write the Hamacher operators in a new form

$$c_H(\mathbf{x}) = \frac{1}{1 + \frac{1}{\gamma_c}\left(\prod_{i=1}^n \left(1 + \gamma_c \frac{1-x_i}{x_i}\right) - 1\right)} \tag{33}$$

and

$$d_H(\mathbf{x}) = \frac{1}{1 + \left(\frac{1}{\gamma_d}\left(\prod_{i=1}^n \left(1 + \gamma_d \frac{x_i}{1-x_i}\right) - 1\right)\right)^{-1}} \tag{34}$$

Comparing the new form of the Hamacher operators (33) and (34) with the originally proposed ones, the new forms look more adequate for several variables. We note that the multi-variable form of this operator did not appear before in the literature.

Using the Generalized Dombi operator then

$$o_\gamma^{(\alpha)}(\mathbf{x}) = \frac{1}{1 + \left(\frac{1}{\gamma}\left(\prod_{i=1}^n \left(1 + \gamma\left(\frac{1-x_i}{x_i}\right)^\alpha\right)\right) - 1\right)^{1/\alpha}} \tag{35}$$

where $\alpha \in \{-1, 1\}$ is a common form for the Hamacher operators. So

$$o_{GD,\gamma_c}^{(1)}(\mathbf{x}) = c_{H,\gamma}(\mathbf{x}) \tag{36}$$

$$o_{GD,\gamma_d}^{(-1)}(\mathbf{x}) = d_{H,\gamma'}(\mathbf{x}) \tag{37}$$

where $\gamma_c, \gamma_d \in [0, \infty)$, $\gamma \in [0, \infty)$ and $\gamma' \in [-1, \infty)$.

VII. THE PRODUCT OPERATOR CASE

The product operator is one of the most widely used in applications of fuzzy sets. Zadeh in his first paper also suggested its use. It is also called probabilistic operator because the probabilities of independent events is the product of the event probabilities. It has the following form:

$$c_P(\mathbf{x}) = \prod_{i=1}^n x_i \tag{38}$$

$$d_P(\mathbf{x}) = 1 - \prod_{i=1}^n (1 - x_i)\,. \tag{39}$$

Because the product operator is a special case of the Hamacher operator the following Theorem holds.

Theorem 9: The product operator is a special case of the Generalized Dombi operator, i.e. if $\gamma = 1$ and $\alpha = \pm 1$,

$$c_{GD,1}^{(1)}(\mathbf{x}) = c_P(\mathbf{x}) \tag{40}$$

$$d_{GD,1}^{(-1)}(\mathbf{x}) = d_P(\mathbf{x}). \tag{41}$$

VIII. THE EINSTEIN OPERATOR CASE

Einstein in his famous work on special relativity theory examined how two velocities have to be added. His result was

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 \cdot v_2}{c^2}}. \tag{42}$$

where v_1 and v_2 are the summands and c is the speed of light. Let us introduce the relative velocities to c as $x = v_1/c$, $y = v_2/c$ and $z = v/c$, then

$$d_E(x, y) = z = \frac{x + y}{1 + xy}. \tag{43}$$

It is easy to check that d_E is a disjunctive operator. Because (43) can be derived from (42) it is called Einstein operator. The corresponding conjunctive operator can be built by using the De Morgan identity with the negation $n(x) = 1 - x$.

Because the Einstein operator is a special case of the Hamacher operator the following Theorem holds.

Theorem 10: The Einstein operator is a special case of the Generalized Dombi operator, i.e. if $\gamma = 2$ and $\alpha = \pm 1$.

$$c_{GD,2}^{(1)}(\mathbf{x}) = c_E(\mathbf{x}) \tag{44}$$

$$d_{GD,2}^{(-1)}(\mathbf{x}) = d_E(\mathbf{x}). \tag{45}$$

Using this result, the n-ary Einstein operators are

$$c_{GD,2}^{(1)}(\mathbf{x}) = \frac{1}{1 + \frac{1}{2}\left(\prod_{i=1}^n \left(1 + 2\frac{1-x_i}{x_i}\right) - 1\right)} \tag{46}$$

$$d_{GD,2}^{(-1)}(\mathbf{x}) = \frac{1}{1 + 2\left(\prod_{i=1}^n \left(1 + 2\frac{x_i}{1-x_i}\right) - 1\right)^{-1}} \tag{47}$$

and we can give the general additivity law of velocities in the framework of special relativity theory.

Corollary 11: Einstein's general additivity law of velocities in his special relativity theory is

$$v = \frac{c}{1 + 2\left(\prod_{i=1}^n \left(1 + 2\frac{v_i}{c - v_i}\right) - 1\right)^{-1}}. \tag{48}$$

Proof. Let us introduce $x_i = v_i/c$, then

$$\frac{x_i}{1 - x_i} = \frac{v_i/c}{1 - v_i/c} = \frac{v_i}{c - v_i}. \tag{49}$$

Using (47) and the substitution, we get (48). ■

IX. THE DRASTIC OPERATOR CASE

The drastic operators introduced by Schweizer and Sklar in 1960 (see [11]) are used if we want to go as close as possible to the two valued logic. Because from the solution of the associativity equation it is known that

$$\begin{aligned} c(x,1) &= c(1,x) = x \\ c(x,0) &= c(0,x) = 0 \\ d(x,1) &= d(1,x) = 1 \\ d(x,0) &= d(0,x) = x \end{aligned}$$

so the drastic operator in the conjunctive case takes the value 0 if $x, y \in [0,1)$ and in the disjunctive case takes the value 1 if $x, y \in (0,1]$. So the drastic operators are

$$c_{Dr}(x,y) = \begin{cases} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0 \text{ otherwise} \end{cases} \tag{50}$$

$$d_{Dr}(x,y) = \begin{cases} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 1 \text{ otherwise} \end{cases} \tag{51}$$

Theorem 12: The drastic operator class is a special case of the Generalized Dombi operator if $\gamma = \infty$.

$$o^{(\alpha)}_{GD,\infty}(\mathbf{x}) = c_{Dr}(\mathbf{x}) \tag{52}$$

$$o^{(-\alpha)}_{GD,\infty}(\mathbf{x}) = d_{Dr}(\mathbf{x}) \tag{53}$$

Proof. The boundary case $c^{(\alpha)}_{GD,\gamma}(x,1,\ldots,1)$ can be checked directly from the formula, and it holds for all γ and α. The case $\forall x_i \in (0,1)$ can be proved as follows. $c^{(\alpha)}_{GD,\gamma}$ can be expanded to

$$o^{(\alpha)}_{GD,\gamma}(\mathbf{x}) = \frac{1}{1 + D^{(\alpha)}_{\gamma}(\mathbf{x})} \tag{54}$$

where

$$\begin{aligned} D^{(\alpha)}_{\gamma}(\mathbf{x}) = \sum_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{\alpha} + \\ + \gamma \sum_{\substack{i,j \in \{1\ldots n\} \\ i \neq j}} \left(\frac{1-x_i}{x_i}\right)^{\alpha} \left(\frac{1-x_j}{x_j}\right)^{\alpha} \\ \ldots \\ + \gamma^n \prod_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{\alpha}. \end{aligned} \tag{55}$$

It is easy to see that

$$\lim_{\gamma \to \infty} D^{(\alpha)}_{\gamma}(\mathbf{x}) = 0 \qquad \text{if } x_i \notin \{0,1\}. \tag{56}$$

for all $\alpha < \infty$, and so

$$\lim_{\gamma \to \infty} c^{(\alpha)}_{GD,\infty}(\mathbf{x}) = 0 \qquad \text{if } x_i \notin \{0,1\}. \tag{57}$$

The disjunctive case can be proved similarly. ■

X. THE MIN AND MAX OPERATOR CASE

The most widely used operators in fuzzy systems are the min and the max. They have many advantages: they are easy to calculate, and they can be extended into a lattice structure. Although in practice the strict operators are more intensively used. The reason is that in the min-max case, the result is determined only by one variable and the other have no influence, opposite to the strictly increasing operators like the Generalized Dombi operator class. Moreover, the min-max operators are not analytical, their second derivative is not continuous. We can establish the following theorem.

Theorem 13: The min and max operators are the limits of strict operators, i.e.

$$\lim_{\alpha \to \infty} c_\alpha(x,y) = \min(x,y), \tag{58}$$

$$\lim_{\alpha \to \infty} d_\alpha(x,y) = \max(x,y). \tag{59}$$

where c_α and d_α are strict operators with generator functions f_c^α and f_d^α $(\alpha > 0)$.

Proof. Let f_c be a generator function of a conjunctive operator c, and $\alpha > 0$. Then, since $f(1) = 0$ implies $f^\alpha(1) = 0$ and since powers do not affect monotonicity, f^α is also a generator function of a conjunctive operator denoted by c_α. Let $x < y$ then

$$\begin{aligned} c_\alpha(x,y) &= f^{-1}\left(\left(f^\alpha(x) + f^\alpha(y)\right)^{1/\alpha}\right) = \\ &= f^{-1}\left(f(x)\left(1 + \frac{f^\alpha(y)}{f^\alpha(x)}\right)^{1/\alpha}\right). \end{aligned} \tag{60}$$

Because $A = f^\alpha(y)/f^\alpha(x) < 1$ and

$$\lim_{\alpha \to \infty} (1 + A^\alpha)^{1/\alpha} = 1 \qquad 0 < A < 1$$

so

$$\lim_{\alpha \to \infty} c_\alpha(x,y) = x = \min(x,y).$$

The theorem is valid for several arguments as well.

$$\lim_{\alpha \to \infty} c_\alpha(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n).$$

A similar proof can be given for disjunctive operators. ■

Corollary 14: The min and max operators are the limits of the Generalized Dombi operator in case $\gamma_c = \gamma_d = 0$, and $\alpha \to \infty$ or $\alpha \to -\infty$, i.e.

$$\lim_{\alpha \to \infty} o^{(\alpha)}_{GD,0}(x,y) = \min(x,y), \tag{61}$$

$$\lim_{\alpha \to -\infty} o^{(\alpha)}_{GD,0}(x,y) = \max(x,y). \tag{62}$$

XI. NEW FORM OF NEGATIONS

In 1977, Sugeno introduced a family of negations [12]:

$$n_\lambda(x) = \frac{1-x}{1+\lambda x}, \qquad \lambda > -1. \tag{63}$$

Note that $n_\lambda = n_\mu$ means $\lambda = \mu$. Observe also $n_0(x) = 1 - x$, and that n satisfies conditions N1-N4 if and only if it is decreasing ($n(x) \geq n(y)$ whenever $x \leq y$) and involutive. In this case we say that n is a *strong negation* (also, a *strict involution*). Obviously, each member of the Sugeno family is a strong negation.

Trillas in 1979 gave a general representation theorem of the negation [13]:

$$n(x)=f^{-1}\left(1-f(x)\right),$$

where $f:[0,1]\to[0,+\infty)$ is a continuous strictly increasing function with $f(0)=0$. According to this, infinitely many negations exist. Hamacher [2] showed that the only rational strong negations are of the form (63). Here we modify (63) and give the semantic meaning of the technical parameter. Two types of characterizations will be given.

Any strong negation has a unique fix point $\nu_*\in(0,1)$ such that

$$n(\nu_*)=\nu_*. \tag{64}$$

If we fix a neutral value $\nu_0\in(0,1)$ (usually it is $\nu_0=1/2$) then there exists a $\nu\in(0,1)$ such that

$$n(\nu)=\nu_0 \tag{65}$$

and λ can be expressed by ν_* or ν and ν_0. In case of rational strong negations we can establish the following.

Theorem 15: Given any $(\nu,\nu_0)\in(0,1)^2$ there exists a unique Sugeno's negation n_λ, such that $n_\lambda(\nu)=\nu_0$. It is defined by

$$\lambda=\frac{1-\nu-\nu_0}{\nu\nu_0},$$

and so

$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}}.$$

In particular, n_λ with

$$\lambda=\frac{1-2\nu_*}{\nu_*^2}$$

is the unique Sugeno's negation having $\nu_*\in(0,1)$ as fix point, i.e.

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}.$$

Proof. Equation (64) means in case of (63) that

$$\nu_*=\frac{1-\nu_*}{1+\lambda\nu_*}.$$

Expressing λ from this we get

$$\lambda=\frac{1-2\nu_*}{\nu_*^2}$$

Substituting λ into (63) we get

$$n_{\nu_*}(x)=\frac{1-x}{1+\frac{1-2\nu_*}{\nu_*^2}x}=\frac{1-x}{1-x+\left(\frac{1-2\nu_*}{\nu_*^2}+1\right)x}=$$
$$=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}$$

and it is the desired form of the negation. Similarly, equation (65) means:

$$\nu_0=\frac{1-\nu}{1+\lambda\nu}.$$

Expressing λ from this:

$$\lambda=\frac{1-\nu-\nu_0}{\nu_0\nu}.$$

Substituting λ into (63) we get

$$n_{\nu,\nu_0}(x)=\frac{1-x}{1+\frac{1-\nu-\nu_0}{\nu_0\nu}x}=$$
$$=\frac{1-x}{1-x+x\left(\frac{1-\nu-\nu_0+\nu_0\nu}{\nu_0\nu}\right)}=$$
$$=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}}.$$

■

XII. THE DE MORGAN LAW IF $\gamma\in(0,\infty)$

It is natural to investigate the validity of the De Morgan law in a consistent logical system w.r.t. some particular negations. In this section we examine the necessary and sufficient conditions of it. We suppose that the conjunctive and the disjunctive operators have the same α. The three operators are:

$$c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x})=\frac{1}{1+D^{(\alpha)}_{\gamma_c}(\mathbf{x})}\qquad \alpha>0 \tag{66}$$

$$d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x})=\frac{1}{1+D^{(\alpha)}_{\gamma_d}(\mathbf{x})}\qquad \alpha<0 \tag{67}$$

where

$$D^{(\alpha)}_{\gamma}(\mathbf{x})=\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}$$

and

$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}} \tag{68}$$

or

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}} \tag{69}$$

where $\gamma_c,\gamma_d\in(0,\infty)$ and $\nu,\nu_0,\nu_*\in(0,1)$.

Theorem 16: The Generalized Dombi operator class (i.e. equations (66), (67) w.r.t the negations of (68),(69)) is a De Morgan triple if and only if

$$\frac{\gamma_d}{\gamma_c}=\left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha} \tag{70}$$

or

$$\frac{\gamma_d}{\gamma_c}=\left(\frac{1-\nu_*}{\nu_*}\right)^{2\alpha} \tag{71}$$

Proof. First we calculate

$$c_{GD}\left(n(x_1),\dots,n(x_n)\right)=c_{GD}\left(n(x)\right)$$

We have to calculate the:

$$\left(\frac{1-n(x)}{n(x)}\right)^{\alpha}=\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^{\alpha} \tag{72}$$

and substitute into the conjunction operator, so we get that $c_{GD}\left(n(x)\right)$ equals

$$\frac{1}{1+\left(\frac{1}{\gamma_c}\left(\prod\left(1+\gamma_c\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{73}$$

Let us calculate $n\left(d_{GD}(\mathbf{x})\right)$. The disjunctive operator has the form

$$d_{GD}(\mathbf{x}) = \frac{1}{1+K}$$

so

$$\begin{aligned} n\left(d_{GD}(\mathbf{x})\right) &= \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-\frac{1}{1+K}}{\frac{1}{1+K}}\right)^{-1}} = \\ &= \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}K^{-1}} \end{aligned}$$

Using this result we get that $n\left(d_{GD}(\mathbf{x})\right)$ equals

$$\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1}{\gamma_d}\left(\prod\left(1+\gamma_d\left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{74}$$

The coefficients of (73) and (74) must be equal before the product

$$\left(\frac{1}{\gamma_c}\right)^{1/\alpha} = \frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1}{\gamma_d}\right)^{1/\alpha} \tag{75}$$

and inside the product

$$\gamma_d = \gamma_c\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\right)^{\alpha}. \tag{76}$$

It is easy to check that the conditions (75) and (76) are equivalent with each other and (70). (71) can be proved in a similar way. ■

Corollary 17: If the negation is

$$n(x) = 1-x$$

i.e. $\nu = \nu_0 = \nu_* = 1/2$ then (66) and (67) form De Morgan triples with $n(x)$ if and only if

$$\gamma = \gamma_c = \gamma_d. \tag{77}$$

Proof. Using (70) we get

$$\frac{\gamma_d}{\gamma_c} = 1.$$

So by using (20) $o_{GD}^{\alpha}(\mathbf{x}) = c_{GD}^{\alpha}(\mathbf{x})$ and $o_{GD}^{-\alpha}(\mathbf{x}) = c_{GD}^{-\alpha}(\mathbf{x})$ and $n(x) = 1-x$ are De Morgan triples. ■

XIII. THE DE MORGAN LAW IF $\gamma = 0$, $\gamma = \infty$ AND $\alpha = \infty$

In Theorem 16 we did not examine the case of $\gamma = 0$ (Dombi operator), $\gamma = 1$ (drastic operator) and $\alpha = \pm\infty$ (min-max operator).

Theorem 18: The Dombi operators form a De Morgan triple with the negations (68) and (69) with the same α for all $\nu_*, \nu, \nu_0 \in (0,1)$.

Proof. Using (72) we get

$$c_D\left(n(\mathbf{x})\right) = \frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^{\alpha}\right)^{1/\alpha}}$$

Similar way as we calculated (74)

$$n\left(d_D(x)\right) = \frac{1}{1+\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\sum_{i=1}^{n}\left(\frac{x}{1-x}\right)^{\alpha}\right)^{1/\alpha}}$$

So $c_D\left(n(\mathbf{x})\right) = n\left(d_D(x)\right)$. ■

Theorem 18 states that by using the Dombi operator class, a wide range of negations can be used to form a De Morgan triplet.

Theorem 19: The min-max operators and the drastic operators are De Morgan triples with any involutive negation. This is a trivial consequence of the fact that min/max and the drastic operators are invariant under any increasing bijections [6]. Now we give a trivial proof of it.

Proof. Let us prove the min-max case first. If $x < y$ then $n(x) > n(y)$ so

$$n(\min(x,y)) = n(n(y)) = y = \max(x,y).$$

If $y < x$ it can be proved similarly.

In the drastic case if $x \in (0,1]$ and $y \in (0,1]$ then $n(x), n(y) \in [0,1)$ and

$$n(c_{Dr}(n(x),n(y))) = n(0) = 1 = d_{Dr}(x,y).$$

If $x = 0$ then $n(x) = 1$ so

$$n(c_{Dr}(1,n(y))) = n(n(y)) = y = d_{Dr}(0,y).$$

■

Corollary 20: The min-max operators form De Morgan triples with the negations (68) and (69) for all $\nu_*, \nu, \nu_0 \in (0,1)$.

XIV. CONCLUSIONS

In this paper we have

1) proved the associativity of the multiplicative utility function,
2) introduced the Generalized Dombi operator:

$$\frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}$$

3) shown new forms of rational involutive negations:

$$n_{\nu_*}(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}}$$

$$n_{\nu,\nu_0}(x) = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}}$$

4) proved that the new operator connectives form a De Morgan triple with a negation iff

$$\frac{\gamma_d}{\gamma_c} = \left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha}$$

5) proved that the Dombi operators form a De Morgan triple with any rational involutive negation
6) shown that the Generalized Dombi operator has the following limits

Type of operator	Value of γ	Value of α	
		conj.	disj.
Dombi	0	$0 < \alpha$	$\alpha < 0$
Product	1	1	-1
Einstein	2	1	-1
Hamacher	$\gamma \in (0, \infty)$	1	-1
Drastic	∞	$0 < \alpha$	$\alpha < 0$
Min-max	0	∞	$-\infty$

7) introduced new forms of the Hamacher operators

$$o_H^{(\alpha)}(\mathbf{x}) = \frac{1}{1 + \left(\frac{1}{\gamma_d}\left(\prod_{i=1}^n \left(1 + \gamma_d \left(\frac{1-x_i}{x_i}\right)^\alpha\right) - 1\right)\right)^{1/\alpha}}$$

8) introduced new forms of the Einstein operators

$$o_{GD,2}^{(\alpha)}(\mathbf{x}) = \frac{1}{1 + 2\left(\prod_{i=1}^n \left(1 + 2\left(\frac{1-x_i}{x_i}\right)^\alpha\right) - 1\right)^{1/\alpha}}$$

9) shown that the addition of several velocities in the framework of special relativity is:

$$v = \frac{c}{1 + 2\left(\prod_{i=1}^n \left(1 + 2\frac{v_i}{c-v_i}\right) - 1\right)^{-1}}.$$

This new parametric operator family is very useful in applications. The two parameters offer more freedom in the sense that by adopting two, instead of just one parameter, the oprator can be fit better to the current problem in investigation.

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