Pliant Ranking

Extended Abstract

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1 Introduction

Multi-criteria decision management applications are important tools for decision makers. One of the difficulties in decision management is the comparison of the alternative choices. The alternatives are often described with human words or given with fuzzy boundaries and as such cannot be ranked easily. Fuzzy theory [10] provides a mathematical foundation for modelling imprecise values and elusive human words [11]. There has already been a lot of effort to aid the decision process with fuzzy ranking algorithms.

One of the earliest fuzzy ranking function comes from Shimura and is based on a comparison method which is adopted for psychological test [9]. Buckley and Chanas [1],[2],[3] provided a fast ranking method using interval analysis. The ranking algorithm introduced by Cheng [4] is based on calculating distances. Delgado et al. [5] gave an ordering procedure using fuzzy relations and fuzzy measures. The preference relation described by Kundu [8] utilizes a fuzzy leftness relation on intervals.

Despite the many different approaches, there is still no consensus which ranking method is the best suitable for applications. The problem arises from the requirements that the ranking algorithm should run fast and have all the properties that a ranking procedure over crisply defined objects has.

In this paper we propose a novel preference ranking method based on the pliant concept [7]. Pliant ranking provides a pairwise choice between two alternatives and can be calculated easily. Our algorithm models the various alternatives with fuzzy numbers and defines a preference relation over them. We prove that pliant ranking fulfills all the necessary properties associated with preference methods, i.e. invariance, transitivity and monotonicity.

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2 Basic definitions

Pliant ranking is an ordering procedure for fuzzy numbers based on preference relations. A fuzzy preference relation is a two-variable fuzzy relation over fuzzy numbers.

Definition 2.1. *Let A and B be fuzzy numbers and $p(x, y)$ a fuzzy preference relation. We say that A is better than B and denote it as*

$$A >_p B \qquad \text{iff.} \qquad p(A, B) > p(A, A) = p(B, B).$$

The value of $p(A, A)$ for any fuzzy number A is called the mean value of the preference. The value of the preference $p(A, B)$ measures how much alternative A is better than alternative B.

In the followings we describe the properties that are considered as important for fuzzy preference relations.

Definition 2.2. *(Shift invariance) Let $p(x, y)$ be a fuzzy preference relation, A and B fuzzy numbers. Let us denote the membership functions of A and B as $\mu_A(x)$ and $\mu_B(x)$ and let $d \in \mathbb{R}$.*

Let A' and B' be defined by $\mu_{A'}(x) = \mu_A(x + d)$ and $\mu_{B'}(x) = \mu_B(x + d)$, i.e. by shifting the fuzzy numbers A and B along the horizontal axis with d. The relation $p(x, y)$ is called shift invariant if $p(A, B) = p(A', B')$.

Definition 2.3. *(Transitivity) Let $p(x, y)$ be a fuzzy preference relation, A, B, and C fuzzy numbers. The relation $p(x, y)$ is transitive if*

$$p(A, B) \geq p(A, A) \quad \text{and} \quad p(B, C) \geq p(B, B), \quad \text{then} \quad p(A, C) \geq p(C, C),$$

where $p(A, A) = p(B, B) = p(C, C)$ is the mean value of the preference.

Definition 2.4. *(Shift monotonicity) Let $p(x, y)$ be a fuzzy preference relation, A and B fuzzy numbers. Let us denote the membership functions of A and B as $\mu_A(x)$ and $\mu_B(x)$.*

The relation $p(x, y)$ is called shift monotone iff.

$$\begin{aligned} \forall d > 0 \in \mathbb{R} &: \quad p(A', B) > p(A, B) \\ \forall d < 0 \in \mathbb{R} &: \quad p(A', B) < p(A, B), \end{aligned}$$

where A' is defined as $\mu_{A'}(x) = \mu_A(x + d)$.

The pliant ranking method is based on the pliant concept and pliant arithmetic operations. Pliant arithmetics is a fuzzy arithmetic framework that defines the arithmetic operations by simple relations between the parameters of the operands. Pliant arithmetics is based on the arithmetic characterization of imprecise quantities with $a < x$ and $x < b$ bounding inequalities. Pliant numbers are created by *softening* the $a < x$ and $x < b$ inequalities i.e., replacing the crisp characteristic function with two fuzzy membership functions and applying a fuzzy conjunction operator. The softened inequalities are referred as *pliant inequalities* and denoted by $[a <_p x]$ and $[x <_p b]$, where p is a control parameter of the fuzzy membership function.

The membership function corresponding to the $a < x$ interval is called the left side of the pliant number. Similarly, the membership function corresponding to the $x < b$ interval is referred as the right side of the pliant number. Pliant operations are carried out by separately applying them to the left sides and to the right sides of the pliant numbers. This decomposition allows us to treat pliant numbers as monotone functions when carrying out pliant arithmetic operations.

Two classes of pliant arithmetics are the additive pliant and the multiplicative pliant. The additive pliant corresponds to the Łukasiewicz operator family and to the linear membership function. The multiplicative pliant corresponds to the Dombi operator family [6] and to the sigmoid membership function.

3 Overview

Piant ranking defines a preference relation to decide whether a pliant number is greater than another. First, the generic pliant preference method is presented. The calculation procedure is based on the idea used in pliant arithmetic operations. The algorithm is then applied to two types of pliant numbers and it is shown that in both cases the preference relation has all the required properties.

Pliant Preference Algorithm 3.1. *Let A, B pliant numbers composed of strictly monotone pliant inequalities. The pliant preference $p(A, B)$, meaning how much A is greater than B is calculated as follows.*

1. *Decompose A and B to their left and right hand side pliant inequalities denoted as A_l, B_l and A_r, B_r respectively.*
2. *Calculate the inverse of the pliant inequalities. Let the inverse functions be denoted as A_l^{-1}, B_l^{-1} and A_r^{-1}, B_r^{-1}.*
3. *Calculate the left hand side preference relation $p_l(A_l^{-1}, B_l^{-1})$ with*

$$p_l(A_l^{-1}, B_l^{-1}) \quad = \quad \frac{1}{1 + e^{(A_l^{-1} - B_l^{-1})}}.$$

   Calculation of the right hand side preference relation $p_r(A_r^{-1}, B_r^{-1})$ is carried out analogously.

4. *Aggregate the left and right hand side preference relations with the aggregation function*

$$a(p_l, p_r) = \frac{1}{1 + \frac{1-p_l}{p_l}\frac{1-p_r}{p_r}} = \frac{1}{1 + e^{(A_l^{-1} - B_l^{-1})} e^{(A_r^{-1} - B_r^{-1})}}.$$

5. *The pliant preference $p(A, B)$ is then calculated by integrating the aggregated preference function over the unit interval,*

$$p(A, B) = \int_0^1 a(p_l(x), p_r(x))dx.$$

Lemma 3.2. *The mean value of the pliant preference given in Algorithm 3.1 is $\frac{1}{2}$.*

Lemma 3.3. *Let A, B be pliant numbers with triangular membership functions. The pliant preference $p(A, B)$ is shift invariant, transitive and shift monotone.*

Lemma 3.4. *Let A, B be pliant numbers with sigmoid membership functions. The pliant preference $p(A, B)$ can be approximated well with a function $\hat{p}(A, B)$ that is shift invariant, transitive and shift monotone.*

The above two classes of fuzzy numbers are used widely in applications.

4 Conclusion

In this paper a novel ranking method that can be used in multi-criteria decision management is introduced. Pliant ranking is based on the pliant preference function. The pliant preference algorithm provides a generic procedure for comparing pliant numbers with various membership functions. It was shown that in the case of linear and sigmoid membership function the pliant preference fulfills the shift invariance, transitivity and shift monotonicity properties.

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