Addition of Sigmoid-Shaped Fuzzy Numbers Using the Dombi operator and Infinite Sum Theorems

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Abstract

The extension principle defines the arithmetic operations on fuzzy numbers. In the extension principle one can use any t-norm for modeling the conjunction operator. It is therefore important to know, which t-norms are consistent with a particular type of fuzzy number. We call a t-norm consistent, if the arithmetic operation is closed. In this paper we investigate the addition of sigmoid and two bell-shaped membership functions. We prove that the addition is closed if the Dombi operator is used.

Keywords: Zadeh's extension principle, Dombi operator,
sigmoid-shaped fuzzy number.

1 Introduction

Many real world applications have to work with imprecise data. Results of measurements, vague statements, flexible constraints can all be a source of inaccurate information. Fuzzy quantities provide a mathematical model for such imprecise quantities and perceptions. The idea that fuzzy quantities could be arithmetically combined according to the laws of fuzzy set theory is due to Zadeh [25]. Soon after, several researchers worked independently along these lines, such as Jain [16], Mizumoto and Tanaka [21], [22], Nahmias [23], Nguyen [24], Dubois and Prade [4]. An overview of fuzzy arithmetics can be found in Dubois and Prade [6], [8], [1]. Several theoretical details and applications can be found e.g., in monographs of Kaufmann and Gupta [17], [18], and Mares [19]. Two special issues of Fuzzy Sets and Systems [7], [12] were also devoted to the topic. The computational approach to linguistic quantifiers is investigated by Zadeh [26], [27].

Arithmetic operations on fuzzy numbers are defined by Zadeh's extension principle [5]. The sum of fuzzy membership functions $\mu_1(x_1)$, $\mu_2(x_2)$, $\ldots$, $\mu_n(x_n)$ can be computed using the extension principle as follows

$$\mu(z) = \sup_{x_1+x_2+\cdots+x_n=z} \min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}. \quad (1)$$

It is possible to replace the min function in (1) with an arbitrary t-norm. This is called the sup-T sum,

$$\mu(z) = \sup_{x_1+x_2+\cdots+x_n=z} T(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)) \quad (2)$$

where $T(x_1, x_2, \ldots, x_n)$ is an arbitrary t-norm.

Equation (2) enables computing a wide variety of fuzzy number sums. Some t-norms are consistent with a specific class of membership functions while others do not. We call a t-norm consistent for addition, if the sup-T sum is a closed arithmetic operation. It is therefore important to know for a particular type of fuzzy numbers, which t-norms satisfy this requirement. Fullér has studied the sup-T sum with triangular fuzzy numbers [9],[10] and in a more general context [11]. These results were developed further and extended by Hong [13],[14],[15] and Mesiar [20].

In this paper we use sigmoid-shaped functions, which appear frequently in natural processes and the Dombi operator. In Section 3 we investigate the sigmoid membership function, which provide a method to construct asymmetrical shaped fuzzy numbers. In Section 4 two bell-shaped membership functions are examined. Bell-shaped functions can be used to model symmetrical shaped fuzzy numbers. With these choices we prove that the addition operation is closed even in the case of infinite summation.

2 Basic definitions

The Dombi operator [2] is

$$c(x_1, x_2, \ldots, x_n) = \frac{1}{1 + \left(\sum_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{\frac{1}{\alpha}}}. \quad (3)$$

For the special case $\alpha = 1$ it is

$$c(x_1, x_2, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^{n} \frac{1-x_i}{x_i}}. \quad (4)$$

In the rest of this paper we refer to this special case as the Dombi operator. The non-associative averaging Dombi operator is

$$\bar{c}(x_1, x_2, \ldots, x_n) = \frac{1}{1 + \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1-x_i}{x_i}\right)}. \tag{5}$$

By substituting t-norms into the sup-T sum we get specific sums, e.g. product-sum, Hamacher-sum [9],[10]. Following this scheme we substitute the Dombi operator into (2) to get the Dombi-sum of fuzzy numbers,

$$\mu(z) = \sup_{x_1+x_2+\cdots+x_n=z} \frac{1}{1 + \sum_{i=1}^{n} \frac{1-\mu_i(x_i)}{\mu_i(x_i)}}. \tag{6}$$

The definition of the averaging Dombi-sum is given similarly,

$$\mu(z) = \sup_{x_1+x_2+\cdots+x_n=z} \frac{1}{1 + \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1-\mu_i(x_i)}{\mu_i(x_i)}\right)}. \tag{7}$$

3 The Dombi-sum of sigmoid functions

In the following two sections the Dombi-sum of sigmoid fuzzy numbers will be calculated. In our research we are working with pliant systems. A multiplicative pliant system is defined by the equation $f_c(x) \cdot f_d(x) = 1$, where f_c and f_d are the generator functions of the conjunction and disjunction operator. The sigmoid function is used as a membership function in multiplicative pliant systems [3], therefore some basic pliant definitions and notation are also introduced here.

Definition 3.1. *The sigmoid function is*

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}} = \{a <_\lambda x\}$$

where a is the mean value, i.e. $\sigma_a^{(\lambda)}(a) = \frac{1}{2}$.
Note: $\{a <_\lambda x\}$ is the pliant notation of the sigmoid function.
The following properties can be seen from Fig. 1

$$a < x \quad then \quad \{a <_\lambda x\} > \frac{1}{2},$$
$$a = x \quad then \quad \{a <_\lambda a\} = \frac{1}{2},$$
$$a > x \quad then \quad \{a <_\lambda x\} < \frac{1}{2}.$$

In arithmetics $a < x$ and $x < b$ inequalities can be used to characterize imprecise quantities. *Pliant numbers* can also be used to represent imprecise

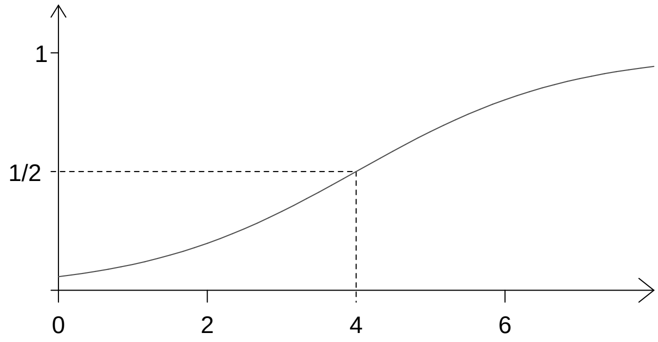

Figure 1: Sigmoid function with $\lambda = 0.7$ and $a = 4$ parameters.

quantities. Pliant numbers are created by *softening* the $a < x$ and $x < b$ inequalities, i.e. replacing the crisp characteristic function with two fuzzy membership functions and applying a fuzzy conjunction operator. The softened inequalities are referred as *pliant inequalities*. Sigmoid functions can be used as pliant inequalities. In this case we represent the imprecise quantity with a sigmoid pliant number,

$$\mu(x) \quad = \quad c\left(\{a <_\lambda x\}, \{b >_\lambda x\}\right)$$

where $c(x_1, x_2)$ is the Dombi operator. The pliant inequalities $\{a <_\lambda x\}$ and $\{b >_\lambda x\}$ are referred as the left side and as the right of the pliant number, respectively.

Arithmetic operations on pliant numbers are executed separately on the left sides and on the right sides of the operands. Combining the left and right side results with the conjunction operator provides the outcome of the operation. Thus if we apply an arithmetic operation to sigmoid functions we need to make sure that the operation is meaningful, i.e. the sigmoid functions represent the same sides. The following criterion formulates this requirement.

Criterion 3.2. *If* $\sigma_{a_1}^{(\lambda_1)}, \sigma_{a_2}^{(\lambda_2)}, \ldots, \sigma_{a_n}^{(\lambda_n)}$ *are inputs to an* n*-ary fuzzy arithmetic operation then*

$$sgn(\lambda_1) = sgn(\lambda_2) = \ldots = sgn(\lambda_n)$$

must always hold.

3.1 Sigmoids with constant lambda value

Let us calculate the sum of $\sigma_{a_1}^{(\lambda)}$,$\sigma_{a_2}^{(\lambda)}$,...,$\sigma_{a_n}^{(\lambda)}$ sigmoid functions, all having the same λ value. The averaging Dombi operator is used so that we have

$$\begin{aligned}\mu(z) &= \sup_{x_1+x_2+\cdots+x_n=z} \bar{c}\left(\sigma_{a_1}^{(\lambda)}(x_1), \sigma_{a_2}^{(\lambda)}(x_2), \ldots, \sigma_{a_n}^{(\lambda)}(x_n)\right) \\ &= \sup_{x_1+x_2+\cdots+x_n=z} \frac{1}{1+\frac{1}{n}\left(\sum_{i=1}^n e^{-\lambda(x_i-a_i)}\right)}. \end{aligned} \tag{8}$$

Theorem 3.3. *The solution of the optimization problem defined in (8) is a sigmoid function*

$$\sigma_{A_n}^{(\Lambda_n)} = \frac{1}{1+e^{-\Lambda_n(x-A_n)}} = \{A_n <_{\Lambda_n} x\} \tag{9}$$

with parameters

$$\Lambda_n = \frac{\lambda}{n} \qquad \textit{and} \qquad A_n = \sum_{i=1}^n a_i.$$

Proof. This constrained optimization problem is solved using the Lagrange multipliers method. The Lagrange function is

$$\Phi(x_1, x_2, \ldots, x_n, \kappa) = \frac{1}{1+\frac{1}{n}\left(\sum_{i=1}^n e^{-\lambda(x_i-a_i)}\right)} + \kappa\left(z - \sum_{i=1}^n x_i\right).$$

We take the partial derivative of Φ with respect to every variable and set the derivatives equal to zero. For every x_i we get

$$\frac{\partial \Phi}{\partial x_i} = \frac{1}{n} \cdot \frac{\lambda e^{-\lambda(x_i-a_i)}}{\left(1+\frac{1}{n}\left(\sum_{i=1}^n e^{-\lambda(x_i-a_i)}\right)\right)^2} - \kappa = 0, \qquad (i = 1, \ldots, n)$$

and for κ we have

$$\frac{\partial \Phi}{\partial \kappa} = z - (x_1 + x_2 + \cdots + x_n) = 0.$$

Let

$$\hat{x}_i = a_i + \frac{z - A_n}{n}.$$

Substituting $\hat{x}_i$ into the system we get

$$\hat{\kappa} = \frac{1}{n} \cdot \frac{\lambda e^{-\lambda\left(\frac{z-A_n}{n}\right)}}{\left(1+e^{-\lambda\left(\frac{z-A_n}{n}\right)}\right)^2}.$$

It can be easily seen that $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{\kappa})$ is a solution of the equation system thus the optimal value of the problem. Substituting the $\hat{x}_i$ values into (8) we get

$$\mu(z) = \frac{1}{1 + \frac{1}{n}\left(\sum_{i=1}^{n} e^{-\lambda(\frac{z-A_n}{n})}\right)} = \frac{1}{1 + e^{-\frac{\lambda}{n}(z-A_n)}}.$$

□

The sum is also a sigmoid function, thus the averaging Dombi-operator is closed under sigmoid addition. The following lemma states an interesting property between the Dombi-sum and the sup-min sum of sigmoid functions.

Lemma 3.4. *The averaging Dombi-sum of* $\sigma_{a_1}^{(\lambda)}$,$\sigma_{a_2}^{(\lambda)}$,…,$\sigma_{a_n}^{(\lambda)}$ *functions is equal with the sup-min sum of the same functions.*

Proof. Dombi showed [3] that the sup-min sum of $\sigma_{a_1}^{(\lambda_1)}$,$\sigma_{a_2}^{(\lambda_2)}$,…,$\sigma_{a_n}^{(\lambda_n)}$ functions is a sigmoid function given as

$$\begin{aligned} \sigma_{A_n}^{(\Lambda_n)} &= \frac{1}{1 + e^{-\Lambda_n(x-A_n)}} \\ &= \sup_{x_1+x_2+\cdots+x_n=z} \min\{\sigma_{a_1}^{(\lambda_1)}(x_1), \sigma_{a_2}^{(\lambda_2)}(x_2), \ldots, \sigma_{a_n}^{(\lambda_n)}(x_n)\}, \end{aligned}$$

where

$$\frac{1}{\Lambda_n} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \quad \text{and} \quad A_n = \sum_{i=1}^{n} a_i.$$

Now by setting the λ_i values equal to λ we get the desired result. □

3.2 Sigmoids with increasing lambda values

The question naturally arises, what happens if the sigmoids have different lambda values. We examine this case by adding an infinite number of sigmoids. As we see this gives us a good condition for choosing the lambda values. First, let a_n be a convergent point sequence. As we approach the limit value of the sequence the points are situated denser and denser as it is illustrated in Fig. 2.

Figure 2: Points of the a_n sequence.

If we were to create triangular fuzzy numbers where each point in the sequence corresponds to the core of one fuzzy number and let the different sides intersect halfway we end up with triangular numbers shown in Fig. 3.

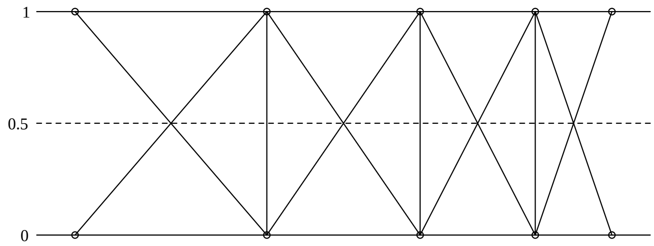

Figure 3: Triangular numbers over the a_n sequence.

As it can be seen the tangent of the lines are increasing as we get closer and closer to the limit value. This is simply because we have chosen the tangent of each line to be equal with $\frac{1}{d_i} = \frac{1}{a_i - a_{i-1}}$, the reciprocal of the distance between two neighbouring points in the sequence. In a sigmoid function the lambda corresponds to the tangent value. Therefore choosing the lambda values similarly we can create an adequate sequence for infinite number of functions. Now let the lambda values be the values of an increasing geometric sequence. The following limit theorem justifies this.

Theorem 3.5. *Let $\sigma_{a_0}^{(\lambda_0)}, \sigma_{a_1}^{(\lambda_1)}, \ldots$ be an infinite series of sigmoid functions where $\lambda_i = \lambda k^i$ with $k > 1$ and $\lambda \neq 0$. If the series $\sum_{i=0}^{\infty} a_i = A$ is convergent, then the Dombi-sum of these functions is a sigmoid function given as*

$$\sigma_B^{(\Lambda)} = \frac{1}{1 + e^{-\Lambda(z-B)}}$$

where

$$\frac{1}{\Lambda} = \frac{k}{\lambda(k-1)} \quad \text{and} \quad B = A - \frac{\frac{k}{k-1} \ln k + k \ln \frac{k}{k-1}}{\lambda(k-1)}. \tag{10}$$

Before proving Theorem 3.5 the Dombi-sum for finite number of sigmoid functions is examined.

Lemma 3.6. *Let $\sigma_{a_0}^{(\lambda_0)}, \sigma_{a_1}^{(\lambda_1)}, \ldots, \sigma_{a_n}^{(\lambda_n)}$ sigmoid functions with $\lambda_i = \lambda k^i$, $k > 1$ and $\lambda \neq 0$. The Dombi-sum of these functions is a sigmoid function given as*

$$\sigma_{B_n}^{(\Lambda_n)} = \frac{1}{1 + e^{-\Lambda_n(z-B_n)}}$$

where

$$\frac{1}{\Lambda_n} = \sum_{i=0}^{n} \frac{1}{\lambda_i} \quad \text{and} \quad B_n = A_n - \frac{(n - G(n)) \ln k + \ln\left(\sum_{i=0}^{n} k^{-i}\right)}{\Lambda_n} \tag{11}$$

using

$$A_n = \sum_{i=0}^{n} a_i \quad \text{and} \quad G_n = \frac{\sum_{i=1}^{n} i k^i}{\sum_{i=0}^{n} k^i}. \tag{12}$$

Proof. The Dombi operator (4) is used, the addition function takes the following form

$$\begin{aligned}\mu(z) &= \sup_{x_0+x_1+\cdots+x_n=z} c\left(\sigma_{a_0}^{(\lambda_0)}(x_0), \sigma_{a_1}^{(\lambda_1)}(x_1), \ldots, \sigma_{a_n}^{(\lambda_n)}(x_n)\right) \\ &= \sup_{x_0+x_1+\cdots+x_n=z} \frac{1}{1+\sum_{i=0}^{n} e^{-\lambda k^i (x_i - a_i)}}.\end{aligned}$$

The Lagrange-multipliers method is used to solve this constrained optimization problem. The Lagrange function is

$$\Phi(x_0, x_1, \ldots, x_n, \kappa) = \frac{1}{1+\sum_{i=0}^{n} e^{-\lambda k^i (x_i - a_i)}} + \kappa\left(z - \sum_{i=0}^{n} x_i\right).$$

We take the partial derivative of Φ with respect to every variable and set the derivatives equal to zero. For every x_i we get

$$\frac{\partial \Phi}{\partial x_i} = \frac{\lambda k^i e^{-\lambda k^i (x_i - a_i)}}{\left(1+\sum_{i=0}^{n} e^{-\lambda k^i (x_i - a_i)}\right)^2} - \kappa = 0, \quad (i = 0, \ldots, n) \tag{13}$$

and for κ we have

$$\frac{\partial \Phi}{\partial \kappa} = z - (x_0 + x_1 + \cdots + x_n) = 0. \tag{14}$$

The values of $x_r (0 \leq r < n)$ are determined as the function of x_n using equation system (13),

$$\begin{aligned}\frac{\lambda k^r e^{-\lambda k^r (x_r - a_r)}}{\left(1+\sum_{i=0}^{n} e^{-\lambda k^i (x_i - a_i)}\right)^2} - \kappa &= \frac{\lambda k^n e^{-\lambda k^n (x_n - a_n)}}{\left(1+\sum_{i=0}^{n} e^{-\lambda k^i (x_i - a_i)}\right)^2} - \kappa, \\ x_r &= -\frac{n-r}{\lambda k^r} \ln k + k^{n-r}(x_n - a_n) + a_r. \quad (15)\end{aligned}$$

We express x_n from (14)

$$x_n = z - \sum_{r=0}^{n-1} x_r,$$

then substitute (15) for all x_r,

$$\begin{aligned}x_n &= z - \sum_{r=0}^{n-1}\left(-\frac{n-r}{\lambda k^r} \ln k + k^{n-r}(x_n - a_n) + a_r\right), \\ x_n &= z + \frac{\ln k}{\lambda} \sum_{r=0}^{n-1} \frac{n-r}{k^r} - x_n \sum_{r=0}^{n-1} k^{n-r} + a_n \sum_{r=0}^{n-1} k^{n-r} - \sum_{r=0}^{n-1} a_r.\end{aligned}$$

Simplifying the exponent of the sum behind x_n and a_n, renaming the index variables, then taking x_n to the left side gives

$$
\begin{aligned}
x_n + x_n \sum_{i=1}^{n} k^i &= z + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i} + a_n \sum_{i=1}^{n} k^i - \sum_{i=0}^{n-1} a_i, \\
x_n &= \frac{z + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i} + a_n \sum_{i=1}^{n} k^i - \sum_{i=0}^{n-1} a_i}{1 + \sum_{i=1}^{n} k^i},
\end{aligned}
$$

and by reindexing the denominator we have

$$
x_n = \frac{z + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i} + a_n \sum_{i=1}^{n} k^i - \sum_{i=0}^{n-1} a_i}{\sum_{i=0}^{n} k^i}.
$$

Finally, a_n is subtracted from both sides and the a_i values are summed on the right hand side,

$$
\begin{aligned}
x_n - a_n &= \frac{z + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i} + a_n \sum_{i=1}^{n} k^i - \sum_{i=0}^{n-1} a_i}{\sum_{i=0}^{n} k^i} - a_n, \\
x_n - a_n &= \frac{z + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i} - \sum_{i=0}^{n} a_i}{\sum_{i=0}^{n} k^i}. \qquad (16)
\end{aligned}
$$

Let us use the definition of $A_n = \sum_{i=0}^{n} a_i$ and substitute (16) into (15) to get the values of x_r $(0 \le r \le n)$,

$$
x_r = -\frac{n-r}{\lambda k^r} \ln k + k^{n-r} \left(\frac{z - A_n + \frac{\ln k}{\lambda} \sum_{i=0}^{n-1} \frac{n-i}{k^i}}{\sum_{i=0}^{n} k^i} \right) + a_r. \qquad (17)
$$

Equation (17) gives the optimal solution to the constrained optimization problem. Equation (18) shows the membership function into which we need to substitute (17),

$$
\mu(z) = \frac{1}{1 + \sum_{r=0}^{n} e^{-\lambda k^r (x_r - a_r)}}. \qquad (18)
$$

Let us do this step by step starting with the exponent and simplifying where it is possible.

$$
\begin{aligned}
x_r - a_r &= -\frac{n-r}{\lambda k^r}\ln k + k^{n-r}\left(\frac{z - A_n + \frac{\ln k}{\lambda}\sum_{i=0}^{n-1}\frac{n-i}{k^i}}{\sum_{i=0}^{n} k^i}\right), \\
-\lambda k^r (x_r - a_r) &= (n-r)\ln k - \lambda k^n \left(\frac{z - A_n + \frac{\ln k}{\lambda}\sum_{i=0}^{n-1}\frac{n-i}{k^i}}{\sum_{i=0}^{n} k^i}\right), \\
-\lambda k^r (x_r - a_r) &= (n-r)\ln k - \frac{\lambda k^n}{\sum_{i=0}^{n} k^i}(z - A_n) \\
&\quad - \frac{\ln k}{\sum_{i=0}^{n} k^i}\sum_{i=0}^{n-1} k^n \frac{n-i}{k^i}, \\
-\lambda k^r (x_r - a_r) &= (n-r)\ln k - \frac{\lambda k^n}{\sum_{i=0}^{n} k^i}(z - A_n) \\
&\quad - \underbrace{\frac{\sum_{i=1}^{n} i k^i}{\sum_{i=0}^{n} k^i}}_{G(n)} \ln k. \qquad (19)
\end{aligned}
$$

Now, let us substitute the λ_i values into Λ_n

$$
\begin{aligned}
\frac{1}{\Lambda_n} &= \sum_{i=0}^{n}\frac{1}{\lambda_i} = \frac{1}{\lambda} + \frac{1}{\lambda k} + \cdots + \frac{1}{\lambda k^n} \qquad (20) \\
&= \frac{k^n + k^{n-1} + \ldots + 1}{\lambda k^n} = \frac{\sum_{i=0}^{n} k^i}{\lambda k^n}.
\end{aligned}
$$

Substituting Λ_n into (19) we have

$$
-\lambda k^r (x_r - a_r) = -\Lambda_n (z - A_n) + (n - G(n) - r)\ln k. \qquad (21)
$$

Substituting (21) into (18) we get the addition function as

$$
\begin{aligned}
\mu(z) &= \frac{1}{1 + e^{-\Lambda_n (z-A_n) + (n-G(n))\ln k}\sum_{r=0}^{n} e^{-r\ln k}} \\
&= \frac{1}{1 + e^{-\Lambda_n (z-A_n) + (n-G(n))\ln k}\sum_{r=0}^{n} k^{-r}} \\
&= \frac{1}{1 + e^{-\Lambda_n (z-A_n) + (n-G(n))\ln k + \ln\left(\sum_{r=0}^{n} k^{-r}\right)}} \\
&= \frac{1}{1 + e^{-\Lambda_n (z-B_n)}}.
\end{aligned}
$$

□

Proof. (Theorem 3.5)
We are looking for the following function

$$\lim_{n\to\infty} \sigma_{B_n}^{(\Lambda_n)} = \frac{1}{1+e^{-\lim_{n\to\infty}\Lambda_n(z-\lim_{n\to\infty}B_n)}}. \tag{22}$$

where Λ_n and B_n are from (10). Let us start with

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} A_n - \lim_{n\to\infty} \frac{(n-G(n))\ln k + \ln\left(\sum_{i=0}^{n} k^{-i}\right)}{\Lambda_n}, \tag{23}$$

because $\lim_{n\to\infty} A_n = A$ we get

$$\lim_{n\to\infty} B_n = A - \frac{\lim_{n\to\infty}(n-G(n))\ln k + \ln\left(\sum_{i=0}^{\infty} k^{-i}\right)}{\lim_{n\to\infty}\Lambda_n}. \tag{24}$$

Since $k > 1$ the geometric series $\sum_{i=0}^{\infty} k^{-i}$ is convergent and the limit of the series is $\frac{k}{k-1}$. By using the definition of Λ_n we get

$$\lim_{n\to\infty} \frac{1}{\Lambda_n} = \frac{1}{\lambda}\sum_{i=0}^{\infty} k^{-i} = \frac{1}{\lambda}\cdot\frac{k}{k-1} = \frac{1}{\Lambda}. \tag{25}$$

Therefore

$$\lim_{n\to\infty} B_n = A - \frac{k\left(\lim_{n\to\infty}(n-G(n))\ln k + \ln\frac{k}{k-1}\right)}{\lambda(k-1)}. \tag{26}$$

Let us now calculate the value of the function $n - G(n)$ where $G(n)$ is given by (12) as

$$G(n) = \frac{\sum_{i=1}^{n} ik^i}{\sum_{i=0}^{n} k^i}.$$

We start by calculating the nominator of $G(n)$, noticing that it can be written as a sum of power function derivatives.

$$\begin{aligned}
\sum_{i=1}^{n} ik^i &= k\sum_{i=1}^{n} ik^{i-1} = k\sum_{i=1}^{n}\left(k^i\right)' = k\left(\sum_{i=1}^{n} k^i\right)' \\
&= k\left(\frac{k-k^{n+1}}{1-k}\right)' \\
&= k\left(\frac{1-(n+1)k^n}{1-k} + \frac{k-k^{n+1}}{(1-k)^2}\right)' \\
&= \frac{nk^{n+2} - nk^{n+1} - k^{n+1} + k}{(1-k)^2}.
\end{aligned}$$

For the denominator of $G(n)$ we get

$$\sum_{i=0}^{n} k^i = \frac{1-k^{n+1}}{1-k} = \frac{k^{n+2}-k^{n+1}-k+1}{(1-k)^2}.$$

Now we get the value of $n - G(n)$ as

$$\begin{aligned} n - G(n) &= n - \frac{nk^{n+2} - nk^{n+1} - k^{n+1} + k}{k^{n+2} - k^{n+1} - k + 1} \\ &= \frac{k^{n+1} - nk + n - k}{k^{n+2} - k^{n+1} - k + 1}. \end{aligned}$$

Now if $k > 1$, then

$$\lim_{n\to\infty} (n - G(n)) = \frac{1}{k-1},$$

which can be easily checked by dividing both the nominator and the denominator with k^{n+1}. This limit is substituted into (26),

$$\begin{aligned} \lim_{n\to\infty} B_n &= A - \frac{k\left(\frac{1}{k-1}\ln k + \ln\frac{k}{k-1}\right)}{\lambda(k-1)} \\ &= A - \frac{\frac{k}{k-1}\ln k + k\ln\frac{k}{k-1}}{\lambda(k-1)} = B. \end{aligned} \tag{27}$$

Finally, we substitute the right sides of (25) and (27) to (22)

$$\lim_{n\to\infty} \sigma_{B_n}^{(\Lambda_n)} = \frac{1}{1+e^{-\Lambda(z-B)}} = \sigma_B^{(\Lambda)}.$$

□

The membership function in Theorem 3.5 can be greatly simplified by choosing a good k constant. Corollary 3.7 sets $k = 2$.

Corollary 3.7. *Let* $\sigma_{a_0}^{(\lambda_0)}, \sigma_{a_1}^{(\lambda_1)}, \ldots$ *be an infinite series of sigmoid functions where* $\lambda_i = \lambda 2^i$ *and* $\lambda \neq 0$*. If the series* $\sum_{i=0}^{\infty} a_i = A$ *is convergent, then the Dombi-sum of these functions is a sigmoid function given as*

$$\sigma_B^{(\Lambda)} = \frac{1}{1+e^{-\Lambda(z-B)}}$$

where

$$\frac{1}{\Lambda} = \frac{2}{\lambda} \quad \textit{and} \quad B = A - \frac{4\ln 2}{\lambda}.$$

4 The Dombi-sum of two classes of bell-shaped numbers

In the following sections the Dombi-sum of two classes of bell-shaped numbers are calculated. The membership function introduced by Zadeh and a membership function based on the powers of two are examined.

Bell-shaped functions are symmetrical sigmoid functions around a center value c and has a width parameter d. The width defines an interval $(c-d, c+d)$, in which the value of the function is greater than a threshold value. Fuzzy numbers with bell-shaped membership functions can be used to model $x \approx c$ arithmetic approximations. This case a fuzzy number is completely defined by one bell-shaped membership function.

Definition 4.1. *The membership function introduced by Zadeh is*

$$\sigma_{c,d}(x) = \frac{1}{1 + \left(\frac{x-c}{d}\right)^2},$$

where c is the center and d is the width of the function.

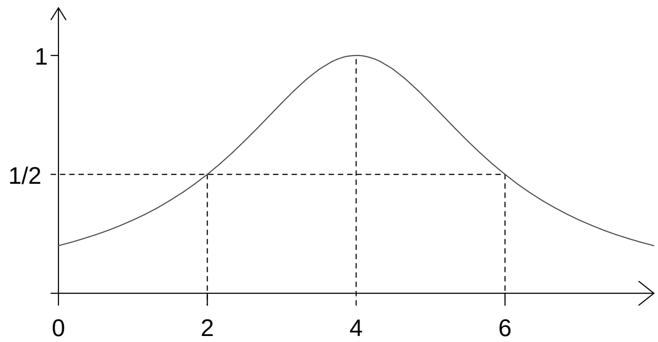

Figure 4: $\sigma_{c,d}(x)$ function with $c = 4$ and $d = 2$ parameters.

The following properties can be seen from Fig. 4

$$\begin{aligned} \sigma_{c,d}(c) &= 1, \\ \sigma_{c,d}(c-d) &= \sigma_{c,d}(c+d) = \frac{1}{2}, \end{aligned}$$

where $\frac{1}{2}$ is the threshold value of the function.

In the next section we calculate the averaging Dombi-sum of fuzzy numbers represented with $\sigma_{c_i,d}(x)$ functions all having the same width parameter.

4.1 Fuzzy numbers with a constant width

The averaging Dombi-sum of $\sigma_{c_1,d}(x), \sigma_{c_2,d}(x), \ldots, \sigma_{c_n,d}(x)$ functions can be given as

$$\begin{aligned}\mu(z) &= \sup_{x_1+x_2+\cdots+x_n=z} \bar{c}\left(\sigma_{c_1,d}(x_1), \sigma_{c_2,d}(x_2), \ldots, \sigma_{c_n,d}(x_n)\right) \\ &= \sup_{x_1+x_2+\cdots+x_n=z} \frac{1}{1+\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{x_i-c_i}{d}\right)^2\right)}. \end{aligned} \tag{28}$$

Theorem 4.2. *The solution of the optimization problem defined in (28) is*

$$\sigma_{C_n,D_n} = \frac{1}{1+\left(\frac{x-C_n}{D_n}\right)^2} \tag{29}$$

with parameters

$$D_n = nd \qquad \textit{and} \qquad C_n = \sum_{i=1}^{n} c_i. \tag{30}$$

Proof. The calculation is analogous to the one presented in Section 3.1. The constrained optimization problem is solved using the Lagrange multipliers method. The corresponding Lagrange function is

$$\Phi(x_1, x_2, \ldots, x_n, \kappa) = \frac{1}{1+\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{x_i-c_i}{d}\right)^2\right)} + \kappa\left(z - \sum_{i=1}^{n} x_i\right).$$

We take the partial derivative of Φ with respect to every variable and set the derivatives equal to zero. For every x_i we get

$$\frac{\partial \Phi}{\partial x_i} = \frac{\frac{2(x_i-c_i)}{nd^2}}{\left(1+\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{x_i-c_i}{d}\right)^2\right)\right)^2} - \kappa = 0,$$

and for κ we have

$$\frac{\partial \Phi}{\partial \kappa} = z - (x_1 + x_2 + \cdots + x_n) = 0.$$

Let

$$\hat{x}_i = c_i + \frac{z - C_n}{n}.$$

Substituting $\hat{x}_i$ into the system we get

$$\hat{\kappa} = \frac{\frac{2(z-C_n)}{n^2 d^2}}{1+\left(\frac{z-C_n}{nd}\right)^2}.$$

It can be easily seen that $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{\kappa})$ is a solution of the equation system thus the optimal value for our conditioned problem. Substituting the $\hat{x}_i$ values into (28) we get

$$\mu(z) = \frac{1}{1+\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{z-C_n}{nd}\right)^2\right)} = \frac{1}{1+\left(\frac{z-C_n}{nd}\right)^2}.$$

□

4.2 Fuzzy numbers with decreasing width

Now let us calculate the sum of an infinite number of fuzzy numbers. In Section 3.2 we reasoned that the tangent of the fuzzy numbers should be increasing. Here, this can be achieved if the width of the numbers are decreasing. More specifically, we use a decreasing geometric sequence.

Theorem 4.3. *Let σ_{c_0,d_0},σ_{c_1,d_1},... be an infinite series of fuzzy numbers with $d_i = dq^i$, $q < 1$ and $d > 0$. If the series $\sum_{i=0}^{\infty} c_i = C$ is convergent, then the Dombi-sum of these functions is*

$$\sigma_{C,D}(z) = \frac{1}{1+\left(\frac{z-C}{D}\right)^2} \quad \textit{where} \quad D = \frac{d}{\sqrt{1-q^2}}.$$

Before proving Theorem 4.3 the Dombi-sum for finite number of fuzzy numbers is calculated.

Lemma 4.4. *Let σ_{c_0,d_0},σ_{c_1,d_1},...,σ_{c_n,d_n} membership functions with $d_i = dq^i$, $q < 1$ and $d > 0$. The Dombi-sum of these functions is*

$$\sigma_{C_n,D_n}(z) = \frac{1}{1+\left(\frac{z-C_n}{D_n}\right)^2}$$

where

$$C_n = \sum_{i=0}^{n} c_i \quad \textit{and} \quad D_n = d\sqrt{\sum_{i=0}^{n} q^{2i}}.$$

Proof. The Dombi operator (4) is used, the addition function takes the following form

$$\begin{aligned}\mu(z) &= \sup_{x_0+x_1+\cdots+x_n=z} c\left(\sigma_{c_0,d_0}(x_0), \sigma_{c_1,d_1}(x_1), \ldots, \sigma_{c_n,d_n}(x_n)\right) \\ &= \sup_{x_0+x_1+\cdots+x_n=z} \frac{1}{1+\sum_{i=0}^{n}\left(\frac{x_i-c_i}{dq^i}\right)^2}.\end{aligned}$$

This constrained optimization problem is solved using the Lagrange multipliers method. The corresponding Lagrange function is

$$\Phi(x_0, x_1, \ldots, x_n, \kappa) = \frac{1}{1+\sum_{i=0}^{n}\left(\frac{x_i-c_i}{dq^i}\right)^2} + \kappa\left(z - \sum_{i=0}^{n} x_i\right).$$

We take the partial derivative of Φ with respect to every variable and set the derivatives equal to zero. For every x_i we get

$$\frac{\partial \Phi}{\partial x_i} = -\frac{\frac{2}{d^2q^{2i}}(x_i-c_i)}{\left(1+\sum_{j=0}^{n}\left(\frac{x_j-c_j}{dq^j}\right)^2\right)^2} - \kappa = 0, \quad (i = 0, \ldots, n) \tag{31}$$

and for κ we have

$$\frac{\partial \Phi}{\partial \kappa} = z - (x_0 + x_1 + \cdots + x_n) = 0. \tag{32}$$

The value $x_r (0 \leq r < n)$ is determined as the function of x_n using the equation system (31).

$$\begin{aligned} -\frac{\frac{2}{d^2q^{2r}}(x_r-c_r)}{\left(1+\sum_{j=0}^{n}\left(\frac{x_j-c_j}{dq^j}\right)^2\right)^2} - \kappa &= -\frac{\frac{2}{d^2q^{2n}}(x_n-c_n)}{\left(1+\sum_{j=0}^{n}\left(\frac{x_j-c_j}{dq^j}\right)^2\right)^2} - \kappa, \\ x_r &= q^{2r-2n}(x_n-c_n)+c_r. \end{aligned} \tag{33}$$

We express x_n from (32)

$$x_n = z - \sum_{r=0}^{n-1} x_r,$$

then substitute (33) for all x_r,

$$
\begin{aligned}
x_n &= z - \sum_{r=0}^{n-1} \left(q^{2r-2n}(x_n - c_n) + c_r\right), \\
x_n &= z - x_n q^{-2n} \sum_{r=0}^{n-1} q^{2r} + c_n q^{-2n} \sum_{r=0}^{n-1} q^{2r} - \sum_{r=0}^{n-1} c_r.
\end{aligned}
$$

Renaming the index variables, then taking x_n to the left side gives

$$
\begin{aligned}
x_n + x_n q^{-2n} \sum_{i=0}^{n-1} q^{2i} &= z + c_n q^{-2n} \sum_{i=0}^{n-1} q^{2i} - \sum_{i=0}^{n-1} c_i, \\
x_n &= \frac{z + c_n q^{-2n} \sum_{i=0}^{n-1} q^{2i} - \sum_{i=0}^{n-1} c_i}{1 + q^{-2n} \sum_{i=0}^{n-1} q^{2i}},
\end{aligned}
$$

and by reindexing the denominator we have

$$
x_n = \frac{z + c_n q^{-2n} \sum_{i=0}^{n-1} q^{2i} - \sum_{i=0}^{n-1} c_i}{q^{-2n} \sum_{i=0}^{n} q^{2i}}.
$$

Finally, c_n is subtracted from both sides and the c_i values are summed on the right hand side,

$$
\begin{aligned}
x_n - c_n &= \frac{z + c_n q^{-2n} \sum_{i=0}^{n-1} q^{2i} - \sum_{i=0}^{n-1} c_i}{q^{-2n} \sum_{i=0}^{n} q^{2i}} - c_n, \\
x_n - c_n &= \frac{z - \sum_{i=0}^{n} c_i}{q^{-2n} \sum_{i=0}^{n} q^{2i}}. \qquad (34)
\end{aligned}
$$

Let us use $C_n = \sum_{i=0}^{n} c_i$ and substitute (34) into (33) to get the value of all x_r $(0 \leq r \leq n)$,

$$
\begin{aligned}
x_r &= q^{2r-2n} \left(\frac{z - C_n}{q^{-2n} \sum_{i=0}^{n} q^{2i}} \right) + c_r, \\
x_r &= q^{2r} \left(\frac{z - C_n}{\sum_{i=0}^{n} q^{2i}} \right) + c_r. \qquad (35)
\end{aligned}
$$

Equation (35) gives the optimal solution to the constrained optimization problem. Equation (36) shows the membership function into which we need to substitute (35),

$$
\mu(z) = \frac{1}{1 + \sum_{r=0}^{n} \left(\frac{x_r - c_r}{d q^r} \right)^2}. \qquad (36)
$$

Let us do this step by step and simplifying where it is possible.

$$\begin{aligned}
x_r - c_r &= q^{2r}\left(\frac{z - C_n}{\sum_{i=0}^{n} q^{2i}}\right), \\
\frac{x_r - c_r}{dq^r} &= q^{r}\left(\frac{z - C_n}{d\sum_{i=0}^{n} q^{2i}}\right), \\
\sum_{r=0}^{n}\left(\frac{x_r - c_r}{dq^r}\right)^2 &= \sum_{r=0}^{n} q^{2r}\left(\frac{z - C_n}{d\sum_{i=0}^{n} q^{2i}}\right)^2, \\
\sum_{r=0}^{n}\left(\frac{x_r - c_r}{dq^r}\right)^2 &= \left(\frac{z - C_n}{d\sqrt{\sum_{i=0}^{n} q^{2i}}}\right)^2. \qquad (37)
\end{aligned}$$

Sudstituting (37) into (36) and using the definition of D_n, we get the addition function as

$$\mu(z) = \frac{1}{1 + \left(\frac{z - C_n}{D_n}\right)^2}. \qquad (38)$$

□

Proof. (Theorem 4.3) Since $q < 1$, the geometric series $\sum_{i=0}^{n} q^{2i}$ is convergent and the limit of the series is $\frac{1}{1-q^2}$. Thus

$$\lim_{n\to\infty} D_n = d\sqrt{\sum_{i=0}^{\infty} q^{2i}} = d \cdot \frac{1}{\sqrt{1-q^2}}.$$

Then by using (38) we have

$$\begin{aligned}
\lim_{n\to\infty} \mu(z) &= \lim_{n\to\infty} \frac{1}{1 + \left(\frac{z - C_n}{D_n}\right)^2} \\
&= \frac{1}{1 + \left(\frac{z - \lim_{n\to\infty} C_n}{\lim_{n\to\infty} D_n}\right)^2} \\
&= \frac{1}{1 + \left(\frac{z - C}{d \cdot \frac{1}{\sqrt{1-q^2}}}\right)^2}
\end{aligned}$$

□

Choosing a good q constant is easy, e.g. let $q = \frac{1}{2}$ so that we get the membership function in Corollary 4.5.

Corollary 4.5. *Let σ_{c_0,d_0},σ_{c_1,d_1},... be an infinite series of Zadeh fuzzy numbers with $d_i = d2^{-i}$, and $d > 0$. If the series $\sum_{i=0}^{\infty} c_i = C$ is convergent, then the Dombi-sum of these functions is a Zadeh fuzzy number given as*

$$\sigma_{C,D}(z) = \frac{1}{1 + \left(\frac{z-C}{D}\right)^2} \quad \textit{where} \quad D = \frac{2d}{\sqrt{3}}.$$

4.3 Bell-shaped function based on the powers of two

We can define a bell-shaped function using only the powers of two as follows.

Definition 4.6. *Let*

$$B_{c,d}(x) = 2^{-\left(\frac{x-c}{d}\right)^2},$$

where c is the center and d is the width of the function.

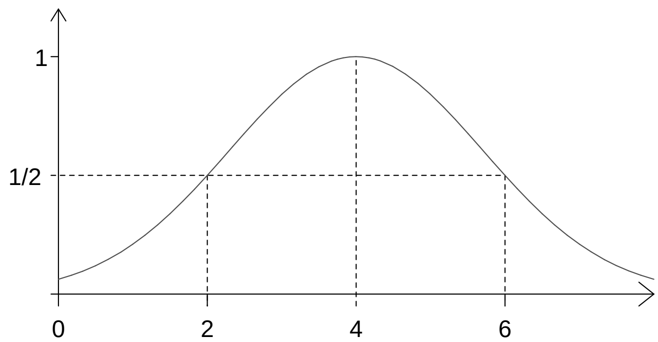

Figure 5: $B_{c,d}(x)$ function with $c = 4$ and $d = 2$ parameters.

The following properties can be seen from Fig. 5

$$\begin{aligned} B_{c,d}(c) &= 1, \\ B_{c,d}(c-d) &= B_{c,d}(c+d) = \frac{1}{2}, \end{aligned}$$

where $\frac{1}{2}$ is the threshold value of the function.

The advantage of this function is that we only have to calculate the powers of a single number. This can be important for implementing fuzzy arithmetic systems on computers. Now, we show that the addition of fuzzy numbers represented with $B_{c_i,d}$ functions is a closed operation.

Theorem 4.7. *The sum of fuzzy numbers $B_{c_1,d_1}, B_{c_2,d_2}, \ldots, B_{c_n,d_n}$ is*

$$B_{C_n,D_n} = 2^{-\left(\frac{x-C_n}{D_n}\right)^2} \tag{39}$$

with parameters

$$D_n = nd \qquad and \qquad C_n = \sum_{i=1}^{n} c_i.$$

Proof. The calculation is analogous to the one presented in Section 4.1. We omit the details here. □

5 Conclusions

In this paper we have presented the Dombi-sum of sigmoid and two bell-shaped functions. It was shown that the addition of fuzzy numbers based on these functions and the Dombi operator is a closed operation. We have also investigated the case when the number of functions are infinite and set a condition on their tangent values. The condition guarantees that the shape of the membership function is preserved also in the case of infinite summation. The results show that the Dombi operator is consistent with sigmoid-shaped functions and the Dombi-sum can be used for addition of fuzzy numbers represented with sigmoid or bell-shaped functions.

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