Towards a Universal Fuzzy Concept: a General Operator Class

József Dombi
Department of Computer Algorithms and Artificial Intelligence
University of Szeged
Szeged, Hungary

***Abstract*— Our starting point is the multiplicative utility function which is extensively used in the theory of multicriteria decision making. Its associativity is shown and as its generalization a fuzzy operator class is introduced with fine and useful properties. As special cases it reduces to well-known operators of fuzzy theory: min/max, product, Einstein, Hamacher, Dombi and drastic. As a consequence, we generalize the addition of velocities in Einstein's special relativity theory to multiple moving objects. Also, a new form of the Hamacher operator is given, which makes multi-argument calculations easier. We examined the De Morgan identity which connects the conjunctive and disjunctive operators by a negation. It is shown that in some special cases (min/max, drastic and Dombi) the operator class forms a De Morgan triple with any involutive negation.**

***Index Terms*— fuzzy operators, hedges, membership function**

I. Introduction

In the last decades many operators were introduced. The most important ones can be found in many handbooks or monographs dealing with fuzzy logic. These operators are: min-max, Hamacher, Einstein, product, Frank, Łukasiewicz, Dombi, Azcél-Alsina, drastic. The Hamacher operator includes the Einstein and the product operators as special cases. The Dombi operator is a special one because the sign of its parameter determines the type of the operator i.e. whether it is conjunctive or disjunctive. From the practical point of view there is a great interest in such parametrial families. There are two main reasons:

- In applications, by changing a single parameter a different logic can be modeled.
- A learning algorithm can find the right type of the operators.

In his first paper on fuzzy sets Zadeh [1] suggested to use the minimum and the product operator. Hamacher [2] discovered that operators can be generated using the solution of the associativity functional equations. Based on the result of Kuwagaki [3], Hamacher got the rational form of conjunctive and disjunctive operators. By that time researchers working on fuzzy theory discovered that they can use a more general framework, i.e. triangular norms. The history of triangular norms started with the paper "Statistical Metric Spaces" by Menger [4]. The terms t-norm and t-conorm were introduced by Schweizer and Sklar [5]. This theory has rather independent roots, namely the theory of functional equations and the theory of groups and semigroups. Several generalizations appeared in this field. A good summarization of the results can be found in Klement et al. [6]. Despite the wide range of operators only a few are used. The main classes of the operators are:

1) min-max, which is widely used in fuzzy theory,
2) drastic operators, which is closest to binary logic,
3) the strict monotone operators, which play an important role in engineering applications. The main strict monotone operators are the product, Hamacher, Frank, Einstein, Dombi operators.
4) the nilpotent operators.

The last one has great importance, because it is very close to the two valued logic. It is associative, the law of contradiction and the law of excluded middle hold, the residual implication can be derived using the connectives, etc. The drawback of this operator class is that its derivative is not continuous, which is important from the practical point of view.

In this article we concentrate on the first three classes of operators. Our main objective is to give a generalized operator class which contains most well-known operators. We define a two-parameter class operator family, which generalize the Dombi operator, preserving its main properties. This operator class contains the Hamacher and Einstein operators, and as a limit we can get the min-max and drastic operators, too.

As a corollary of the multivariate Einstein operator we get the closed form of the additivity law of velocities in the framework of special relativity theory. We give a new form of the Hamacher operator family, with which its multivariate case can be handled more easily. The De Morgan law is also studied and we give a necessary and sufficient condition for it.

II. The Multiplicative Utility Function

In their seminal treatment of multiattribute utility (MAU) theory, Keeney and Raiffa [7] show how certain conditions of independence among attributes yield the so called multiplicative multiattribute utility function

$$u_M(\mathbf{x}) = \frac{1}{k}\left(\prod_{i=1}^{n}\left(1 + kk_iu_i(x_i)\right) - 1\right) \tag{1}$$

which can also be expanded as

$$\begin{aligned} u_M(\mathbf{x}) = &\sum_{i=1}^{n} k_iu_i(x_i) + k\sum_{i<j} k_ik_ju_i(x_i)u_j(x_j) + \\ &+ k^2\sum k_ik_jk_lu_i(x_i)u_j(x_j)u_l(x_l) + \ldots \\ &+ k^{n-1}k_1k_2\ldots k_nu_1(x_1)\ldots u_n(x_n) \end{aligned} \tag{2}$$

where from the normalization of $u_M(\mathbf{x})$ we get

$$1+k=\prod_{i=1}^{n}(1+kk_i), \tag{3}$$

i.e. k is determined only by the weights k_i.

A. The Associativity of the Multiplicative Utility Function

First we show that the multiplicative utility function is associative.

Theorem 1: The multiplicative utility function is associative.

Proof. From the Theorem of Aczel [8] we know that the solution of the associativity functional equation has the form

$$F(\mathbf{x})=f^{-1}\left(\sum f(x_i)\right). \tag{4}$$

We show that the multiplicative utility function can also be written in this form, where

$$f(x)=\ln(1+kx), \tag{5}$$

and

$$f^{-1}(x)=\frac{1}{k}\left(e^x-1\right). \tag{6}$$

with the substitution $x_i := w_i u_i(x_i)$. ■

Lemma 2: As a special case if $k=0$ then

$$u(x)=\sum_i w_i u_i(x_i) \tag{7}$$

Proof. Because

$$\begin{aligned}&\frac{1}{k}\left(\prod_{i=1}^{n}\left(1+kk_iu_i(x_i)\right)-1\right)\\ &=\sum_{i=1}^{n}k_iu_i(x_i)+k\sum_{i<j}k_ik_ju_i(x_i)u_j(x_j)+...\end{aligned} \tag{8}$$

substituting $k=0$ we get the result. ■

B. Logical operators and the Multiplicative Utility Function

Let

$$f(x)=\ln(1+kg(x)) \tag{9}$$

then

$$f^{-1}(x)=g^{-1}\left(\frac{1}{k}e^x-1\right). \tag{10}$$

The associative operator generated by f is

$$o(x_1,\dots,x_n)=g^{-1}\left(\frac{1}{k}\left(\prod\left(1+kg(x_i)\right)-1\right)\right). \tag{11}$$

Similarly to the Lemma above it is valid

$$o(x_1,\dots,x_n)|_{k=0}=g^{-1}\left(\sum g(x_i)\right). \tag{12}$$

It is easy to get logical operators from the above mentioned generalization, i.e. let g be the generator function of a logical operator and substitute it into

$$f(x)=\ln(1+kg(x)). \tag{13}$$

This way we get a logical operator by choosing $k=0$, and $g(x)$ is the generator function of the operator. Next, let $g(x)=\left(\frac{1-x}{x}\right)^{\alpha}$, i.e. the generator function of the Dombi operator.

III. The Generalized Dombi operator

Definition 3: The generator functions of the Generalized Dombi operator are

$$f_c(x)=\ln\left(1+\gamma_c\left(\frac{1-x}{x}\right)^{\alpha}\right) \qquad \alpha>0 \tag{14}$$

$$f_d(x)=\ln\left(1+\gamma_d\left(\frac{1-x}{x}\right)^{\alpha}\right) \qquad \alpha<0 \tag{15}$$

where $\gamma_c,\gamma_d\in[0,\infty]$. From

$$c(\mathbf{x})=f_c^{-1}\left(\sum_{i=1}^{n}f_c(x_i)\right)$$

$$d(\mathbf{x})=f_d^{-1}\left(\sum_{i=1}^{n}f_d(x_i)\right)$$

and

$$f_c^{-1}(x)=\frac{1}{1+\left(\frac{1}{\gamma_c}\left(e^x-1\right)\right)^{1/\alpha}} \qquad \alpha>0 \tag{16}$$

$$f_d^{-1}(x)=\frac{1}{1+\left(\frac{1}{\gamma_d}\left(e^x-1\right)\right)^{1/\alpha}} \qquad \alpha<0 \tag{17}$$

the operators are

$$c_{GD,\gamma_c}^{(\alpha)}(\mathbf{x})=\frac{1}{1+D_{\gamma_c}(\mathbf{x})} \qquad \alpha>0 \tag{18}$$

$$d_{GD,\gamma_d}^{(\alpha)}(\mathbf{x})=\frac{1}{1+D_{\gamma_d}(\mathbf{x})} \qquad \alpha<0 \tag{19}$$

where

$$D_{\gamma}(\mathbf{x})=\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha} \tag{20}$$

and $\gamma_c,\gamma_d\in[0,\infty]$.

It is easy to check that $c_{GD,\gamma_c}^{(\alpha)}(\mathbf{x})$ and $d_{GD,\gamma_d}^{(\alpha)}(\mathbf{x})$ are strict conjunctive and disjunctive operators. Equations (18) and (19) differ only in the sign of α and so the Generalized Dombi operator class is:

$$o_{GD,\gamma}^{(\alpha)}(\mathbf{x})=\frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{21}$$

IV. The Dombi operator case

The Dombi operator has the form (see [9])

$$o_D^{(\alpha)}(\mathbf{x})=\frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \tag{22}$$

and if $\alpha>0$ then the operator is conjunctive and if $\alpha<0$ then the operator is disjunctive. The next Theorem follows from Lemma 2, by the substitution $k=\gamma$.

Theorem 4: The Dombi operator is a special case of the Generalized Dombi operator, i.e. if $\gamma=0$,

$$c_{GD,0}^{(\alpha)}(\mathbf{x})=c_D^{(\alpha)}(\mathbf{x}) \tag{23}$$

$$d_{GD,0}^{(\alpha)}(\mathbf{x})=d_D^{(\alpha)}(\mathbf{x}). \tag{24}$$

V. The Hamacher operator case

Hamacher [2] was one of the first who discussed how new logical operators can be generated using the solutions of the associativity functional equation. As we have seen, with the help of the generator function of the operator, infinitely many operators can be constructed. To restrict the solution space Hamacher added a new requirement, namely he looked for operators which can be written in rational form (a quotient of two polinoms). Kuwagaki [3] showed that in this case the generator function can only have the following two forms:

$$f^{-1}(x) = \frac{ax+b}{cx+d} \quad \text{or} \quad f^{-1}(x) = \frac{ae^x+b}{ce^x+d} \tag{25}$$

Hamacher showed that to have conjunctive or disjunctive operators equations (25) have the following solutions

$$f_c^{-1}(x) = \frac{e^x}{\gamma + (1-\gamma)e^x} \tag{26}$$

$$f_d^{-1}(x) = \frac{e^x - 1}{\gamma' + e^x} \tag{27}$$

The Hamacher operators are

$$c_{H,\gamma}(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)} \tag{28}$$

$$d_{H,\gamma'}(x,y) = \frac{x+y-(1-\gamma')xy}{1+\gamma' xy} \tag{29}$$

where $0 \le \gamma$ and $-1 \le \gamma'$.

Theorem 5: The Hamacher operator class is a special case of the Generalized Dombi operator if $\alpha = \pm 1$ and $\gamma \in (0, \infty)$.

$$c^{(1)}_{GD,\gamma_c}(\mathbf{x}) = c_{H,\gamma}(\mathbf{x}) \tag{30}$$

$$d^{(-1)}_{GD,\gamma_d}(\mathbf{x}) = d_{H,\gamma'}(\mathbf{x}). \tag{31}$$

Proof. The inverse generator function of the Hamacher operator in the conjunctive case can be transformed

$$f_c^{-1}(x) = \frac{e^x}{\gamma + (1-\gamma)e^x} = \frac{1}{\gamma e^{-x} + (1-\gamma)} = \\ = \frac{1}{1+\gamma(e^{-x}-1)}.$$

$f_c^{-1}(x)$ generate the same operator as $f_c^{-1}(Ax)$ where $A \neq 0$ is constant. Let us choose $A = -1$ and $\gamma = 1/\gamma_c$, so

$$f_c^{-1}(x) = \frac{1}{1 + \frac{1}{\gamma_c}(e^x - 1)} \tag{32}$$

which is the same as the inverse generator function of Generalized Dombi operator in the $\alpha = 1$ case.

The inverse generator function of the disjunctive Hamacher operator can be transformed similarly

$$f_d^{-1}(x) = \frac{e^x - 1}{\gamma' + e^x} = \frac{e^x - 1}{e^x - 1 + \gamma' + 1} = \\ = \frac{1}{1 + (\gamma'+1)\left(e^x - 1\right)^{-1}}.$$

Let us choose $\gamma' + 1 = 1/\gamma_d$, so

$$f_c^{-1}(x) = \frac{1}{1 + \frac{1}{\gamma_d}\left(e^x - 1\right)^{-1}} \tag{33}$$

which is the same as the inverse generator function of the Generalized Dombi operator in the $\alpha = -1$ case.

It is easy to see that $0 < \gamma$ and $-1 < \gamma'$ are the same requirements as $\gamma_c \in (0, \infty)$ and $\gamma_d \in (0, \infty)$. ■

Corollary 6: Using the new type of generator functions we can write the Hamacher operators in a new form

$$c_H(\mathbf{x}) = \frac{1}{1 + \frac{1}{\gamma_c}\left(\prod_{i=1}^n \left(1 + \gamma_c \frac{1-x_i}{x_i}\right) - 1\right)} \tag{34}$$

and

$$d_H(\mathbf{x}) = \frac{1}{1 + \left(\frac{1}{\gamma_d}\left(\prod_{i=1}^n \left(1 + \gamma_d \frac{x_i}{1-x_i}\right) - 1\right)\right)^{-1}} \tag{35}$$

Comparing the new form of the Hamacher operators (34) and (35) with the originally proposed ones, the new forms look more adequate for several variables. We note that the multi-variable form of this operator did not appear before in the literature.

Using the Generalized Dombi operator then

$$o_\gamma^{(\alpha)}(\mathbf{x}) = \frac{1}{1 + \left(\frac{1}{\gamma}\left(\prod_{i=1}^n \left(1 + \gamma\left(\frac{1-x_i}{x_i}\right)^\alpha\right)\right) - 1\right)^{1/\alpha}} \tag{36}$$

where $\alpha \in \{-1, 1\}$ is a common form for the Hamacher operators. So

$$o^{(1)}_{GD,\gamma_c}(\mathbf{x}) = c_{H,\gamma}(\mathbf{x}) \tag{37}$$

$$o^{(-1)}_{GD,\gamma_d}(\mathbf{x}) = d_{H,\gamma'}(\mathbf{x}) \tag{38}$$

where $\gamma_c, \gamma_d \in [0, \infty)$, $\gamma \in [0, \infty)$ and $\gamma' \in [-1, \infty)$.

VI. The product operator case

The product operator is one of the most widely used in applications of fuzzy sets. Zadeh in his first paper also suggested its use. It is also called probabilistic operator because the probabilities of independent events is the product of the event probabilities. It has the following form:

$$c_P(\mathbf{x}) = \prod_{i=1}^n x_i \tag{39}$$

$$d_P(\mathbf{x}) = 1 - \prod_{i=1}^n (1 - x_i)\,. \tag{40}$$

Because the product operator is a special case of the Hamacher operator the following Theorem holds.

Theorem 7: The product operator is a special case of the Generalized Dombi operator, i.e. if $\gamma = 1$ and $\alpha = \pm 1$,

$$c^{(1)}_{GD,1}(\mathbf{x}) = c_P(\mathbf{x}) \tag{41}$$

$$d^{(-1)}_{GD,1}(\mathbf{x}) = d_P(\mathbf{x}). \tag{42}$$

VII. The Einstein operator case

Einstein in his famous work on special relativity theory examined how two velocities have to be added. His result was

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 \cdot v_2}{c^2}}. \tag{43}$$

where v_1 and v_2 are the summands and c is the speed of light. Let us introduce the relative velocities to c as $x = v_1/c$, $y = v_2/c$ and $z = v/c$, then

$$d_E(x, y) = z = \frac{x + y}{1 + xy}. \tag{44}$$

It is easy to check that d_E is a disjunctive operator. Because (44) can be derived from (43) it is called Einstein operator. The corresponding conjunctive operator can be built by using the De Morgan identity with the negation $n(x) = 1 - x$.

Because the Einstein operator is a special case of the Hamacher operator the following Theorem holds.

Theorem 8: The Einstein operator is a special case of the Generalized Dombi operator, i.e. if $\gamma = 2$ and $\alpha = \pm 1$.

$$c^{(1)}_{GD,2}(\mathbf{x}) = c_E(\mathbf{x}) \tag{45}$$

$$d^{(-1)}_{GD,2}(\mathbf{x}) = d_E(\mathbf{x}). \tag{46}$$

Using this result, the n-ary Einstein operators are

$$c^{(1)}_{GD,2}(\mathbf{x}) = \frac{1}{1 + \frac{1}{2}\left(\prod_{i=1}^{n}\left(1 + 2\frac{1-x_i}{x_i}\right) - 1\right)} \tag{47}$$

$$d^{(-1)}_{GD,2}(\mathbf{x}) = \frac{1}{1 + 2\left(\prod_{i=1}^{n}\left(1 + 2\frac{x_i}{1-x_i}\right) - 1\right)^{-1}} \tag{48}$$

and we can give the general additivity law of velocities in the framework of special relativity theory.

Corollary 9: Einstein's general additivity law of velocities in his special relativity theory is

$$v = \frac{c}{1 + 2\left(\prod_{i=1}^{n}\left(1 + 2\frac{v_i}{c-v_i}\right) - 1\right)^{-1}}. \tag{49}$$

Proof. Let us introduce $x_i = v_i/c$, then

$$\frac{x_i}{1 - x_i} = \frac{v_i/c}{1 - v_i/c} = \frac{v_i}{c - v_i}. \tag{50}$$

Using (48) and the substitution, we get (49). ■

VIII. The Drastic Operator Case

The drastic operators introduced by Schweizer and Sklar in 1960 (see [10]) are used if we want to go as close as possible to the two valued logic. Because from the solution of the associativity equation it is known that

$$c(x, 1) = c(1, x) = x$$
$$c(x, 0) = c(0, x) = 0$$
$$d(x, 1) = d(1, x) = 1$$
$$d(x, 0) = d(0, x) = x$$

so the drastic operator in the conjunctive case takes the value 0 if $x, y \in [0, 1)$ and in the disjunctive case takes the value 1 if $x, y \in (0, 1]$. So the drastic operators are

$$c_{Dr}(x, y) = \begin{cases} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0 \text{ otherwise} \end{cases} \tag{51}$$

$$d_{Dr}(x, y) = \begin{cases} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 1 \text{ otherwise} \end{cases} \tag{52}$$

Theorem 10: The drastic operator class is a special case of the Generalized Dombi operator if $\gamma = \infty$.

$$o^{(\alpha)}_{GD,\infty}(\mathbf{x}) = c_{Dr}(\mathbf{x}) \tag{53}$$

$$o^{(-\alpha)}_{GD,\infty}(\mathbf{x}) = d_{Dr}(\mathbf{x}) \tag{54}$$

IX. The Min and Max Operator Case

The most widely used operators in fuzzy theory are the min and the max operator. They have many advantages: they are easy to calculate, and they can be extended into a lattice structure. Although in practice the strict operators are more intensively used. The reason is that in the min-max case the result is determined only by one variable and the other have no influence, opposite to the strict monotonously increasing operators like the Generalized Dombi operator class. Moreover, the min-max operators are not analytical, their second derivative is not continuous. We show here that as a limit we can get the min and max operators.

Theorem 11: The min and max operators are the limits of the Generalized Dombi operator if $\gamma = 0$ and $\alpha \to \infty$ or $\alpha \to -\infty$.

$$\lim_{\alpha\to\infty} o^{(\alpha)}_{GD,0}(x, y) = \min(x, y) \tag{55}$$

$$\lim_{\alpha\to-\infty} o^{(\alpha)}_{GD,0}(x, y) = \max(x, y) \tag{56}$$

Proof. A more general theorem is valid. Let $f(x)$ be a generator function of $c(x, y)$ then $f^{(\alpha)}(x)$ is also a generator function of a conjunctive operator denoted by $c_\alpha(x, y)$. We state that

$$\lim_{\alpha\to\infty} c_\alpha(x, y) = \min(x, y) \tag{57}$$

(A similar generalization is valid for the disjunctive operator.) Let $x < y$ then

$$\begin{aligned} c_\alpha(x, y) &= f^{-1}\left(\left(f^\alpha(x) + f^\alpha(y)\right)^{1/\alpha}\right) = \\ &= f^{-1}\left(f(x)\left(1 + \frac{f^\alpha(y)}{f^\alpha(x)}\right)^{1/\alpha}\right). \end{aligned} \tag{58}$$

Because $A = f^\alpha(y)/f^\alpha(x) < 1$ and

$$\lim_{\alpha\to\infty} (1 + A^\alpha)^{1/\alpha} = 1 \qquad 0 < A < 1$$

so

$$\lim_{\alpha\to\infty} c_\alpha(x, y) = x = \min(x, y).$$

The theorem is valid for several arguments as well.

$$\lim_{\alpha\to\infty} c_\alpha(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n).$$

The generator function of the Dombi operator is

$$f^\alpha(x) = \left(\frac{1 - x}{x}\right)^\alpha$$

so the theorem is valid. A similar proof can be given for the disjunctive case. ■

X. New Form of Negations

The negation is a unary operator. Its most frequently used form is

$$n(x) = 1 - x.$$

Sugeno [11] in 1977 also introduced a negation:

$$n(x) = \frac{1-x}{1+\lambda x} \qquad \lambda > -1. \tag{59}$$

The usual requirements for a negation are:

1) $n : [0,1] \to [0,1]$ is continuous
2) strictly decreasing
3) $n(0) = 1$ and $n(1) = 0$
4) $n(x)$ is involutive i.e. $n(n(x)) = x$

Trillas gave the general representation theorem of the negation:

$$n(x) = f^{-1}(1 - f(x))$$

where $f(x)$ is a continuous and strictly increasing function, $f(0) = 0$ and $f(1) = 1$. According to this infinitely many negations exist. Hamacher in his work showed that the rational form of involutive negations is (59) (it is also called Hamacher negation). In this paper we modify the form of (59) and we give the semantic meaning of the technical parameter. Two types of characterizations will be given.

If the negation fulfills axioms (1)-(4) then there exists a fix point ν'_* such that

$$n(\nu_*) = \nu_*. \tag{60}$$

If we fix a neutral value (usually it is $\nu_0 = 1/2$) then there exists a ν value such that

$$n(\nu) = \nu_0 \tag{61}$$

and λ can be expressed by ν_* and $\nu,\ \nu_0$.

Theorem 12: The Dombi form of the negation is

$$n_{\nu_*}(x) = \frac{1}{1 + \left(\frac{1-\nu_*}{\nu_*}\right)^2 \left(\frac{1-x}{x}\right)^{-1}}$$

$$n_{\nu,\nu_0}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}}$$

Proof. Equation (60) means in case of (59) that

$$\nu_* = \frac{1-\nu_*}{1+\lambda\nu_*}.$$

Expressing λ from this we get

$$\lambda = \frac{1-2\nu_*}{\nu_*^2}$$

Substituting λ into (59) we get

$$n_{\nu_*}(x) = \frac{1-x}{1+\frac{1-2\nu_*}{\nu_*^2}x} = \frac{1-x}{1-x+\left(\frac{1-2\nu_*}{\nu_*^2}+1\right)x} = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}$$

and it is the desired form of the negation. Similarly, equation (61) means:

$$\nu_0 = \frac{1-\nu}{1+\lambda\nu}.$$

Expressing λ from this:

$$\lambda = \frac{1-\nu-\nu_0}{\nu_0\nu}.$$

Substituting λ into (59) we get

$$n_{\nu,\nu_0}(x) = \frac{1-x}{1+\frac{1-\nu-\nu_0}{\nu_0\nu}x} = \frac{1-x}{1-x+x\left(\frac{1-\nu-\nu_0+\nu_0\nu}{\nu_0\nu}\right)} = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}}$$

and it is the desired form of the negation. ■

XI. The De Morgan law if $\gamma \in (0,\infty)$

It is natural to demand the validity of the De Morgan law in a consistent logical system. In this section we examine the necessary and sufficient conditions of it. We suppose that the conjunctive and the disjunctive operators have the same α. The three operators are:

$$c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x}) = \frac{1}{1+D_{\gamma_c}(\mathbf{x})} \qquad \alpha > 0 \tag{62}$$

$$d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x}) = \frac{1}{1+D_{\gamma_d}(\mathbf{x})} \qquad \alpha < 0 \tag{63}$$

where

$$D_\gamma(\mathbf{x}) = \left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^\alpha\right)-1\right)\right)^{1/\alpha}$$

and

$$n_{\nu,\nu_0}(x) = \frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}} \tag{64}$$

or

$$n_{\nu_*}(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}} \tag{65}$$

where $\gamma_c, \gamma_d \in (0,\infty)$ and $\nu, \nu_0, \nu_* \in (0,1)$.

Theorem 13: The Generalized Dombi operator class (i.e. equations (62), (63) and (64),(65)) is a De Morgan triple if and only if

$$\frac{\gamma_d}{\gamma_c} = \left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^\alpha \tag{66}$$

or

$$\frac{\gamma_d}{\gamma_c} = \left(\frac{1-\nu_*}{\nu_*}\right)^{2\alpha} \tag{67}$$

Proof. First we calculate

$$c_{GD}(n(x_1),\ldots,n(x_n)) = c_{GD}(\underline{n}(x))$$

We have to calculate the:

$$\left(\frac{1-n(x)}{n(x)}\right)^\alpha = \left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^\alpha \tag{68}$$

and substitute into the conjunction operator, so we get that $c_{GD}\left(\underline{n}(x)\right)$ equals

$$\frac{1}{1+\left(\frac{1}{\gamma_c}\left(\prod\left(1+\gamma_c\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{69}$$

Let us calculate $n\left(d_{GD}(\mathbf{x})\right)$. The disjunctive operator has the form

$$d_{GD}(\mathbf{x})=\frac{1}{1+K}$$

so

$$n\left(d_{GD}(\mathbf{x})\right)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-\frac{1}{1+K}}{\frac{1}{1+K}}\right)^{-1}}=$$
$$=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}K^{-1}}$$

Using this result we get that $n\left(d_{GD}(\mathbf{x})\right)$ equals

$$\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1}{\gamma_d}\left(\prod\left(1+\gamma_d\left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)-1\right)\right)^{1/\alpha}} \tag{70}$$

The coefficients of (69) and (70) must be equal before the product

$$\left(\frac{1}{\gamma_c}\right)^{1/\alpha}=\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1}{\gamma_d}\right)^{1/\alpha} \tag{71}$$

and inside the product

$$\gamma_d=\gamma_c\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\right)^{\alpha}. \tag{72}$$

It is easy to check that the conditions (71) and (72) are equivalent with each other and (66). (67) can be proved in a similar way. ■

Corollary 14: If the negation is

$$n(x)=1-x$$

i.e. $\nu=\nu_0=\nu_*=1/2$ then (62) and (63) form De Morgan triples with $n(x)$ if and only if

$$\gamma=\gamma_c=\gamma_d. \tag{73}$$

Proof. Using (66) we get

$$\frac{\gamma_d}{\gamma_c}=1.$$

So by using (21) $o_{GD}^{\alpha}(\mathbf{x})=c_{GD}^{\alpha}(\mathbf{x})$ and $o_{GD}^{-\alpha}(\mathbf{x})=c_{GD}^{-\alpha}(\mathbf{x})$ and $n(x)=1-x$ are De Morgan triples. ■

XII. The De Morgan law if $\gamma=0$, $\gamma=\infty$ and $\alpha=\infty$

In Theorem 13 we did not examine the case of $\gamma=0$ (Dombi operator), $\gamma=1$ (drastic operator) and $\alpha=\pm\infty$ (min-max operator).

Theorem 15: The Dombi operators form a De Morgan triple with the negations (64) and (65) with the same α for all $\nu_*,\nu,\nu_0\in(0,1)$.

Proof. Using (68) we get

$$c_D\left(\underline{n}(\mathbf{x})\right)=\frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\frac{x}{1-x}\right)^{\alpha}\right)^{1/\alpha}}$$

Similar way as we calculated (70)

$$n\left(d_D(x)\right)=\frac{1}{1+\left(\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\sum_{i=1}^{n}\left(\frac{x}{1-x}\right)^{\alpha}\right)^{1/\alpha}}$$

So $c_D\left(\underline{n}(\mathbf{x})\right)=n\left(d_D(x)\right)$. ■

Theorem 15 states that by using the Dombi operator class, a wide range of negations can be used to form a De Morgan triplet.

Theorem 16: The min-max operators and the drastic operator are De Morgan triples with any involutive negation.

Proof. Let us prove the min-max case first. If $x<y$ then $n(x)>n(y)$ so

$$n(\min(x,y))=n(n(y))=y=\max(x,y).$$

If $y<x$ it can be proved similarly.

In the drastic case if $x\in(0,1]$ and $y\in(0,1]$ then $n(x),n(y)\in[0,1)$ and

$$n(c_{Dr}(n(x),n(y)))=n(0)=1=d_{Dr}(x,y).$$

If $x=0$ then $n(x)=1$ so

$$n(c_{Dr}(1,n(y)))=n(n(y))=y=d_{Dr}(0,y).$$

■

Corollary 17: The min-max operators form De Morgan triples with the negations (64) and (65) for all $\nu_*,\nu,\nu_0\in(0,1)$.

XIII. Conclusions

In this paper we have

1) proved the associativity of the multiplicative utility function,
2) introduced the Generalized Dombi operator:

$$\frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}$$

3) shown new forms of rational involutive negations:

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}}$$
$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}}$$

4) proved that the new operator connectives form a De Morgan triple with a negation iff

$$\frac{\gamma_d}{\gamma_c}=\left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha}$$

5) proved that the Dombi operators form a De Morgan triple with any rational involutive negation
6) shown that the Generalized Dombi operator has the following limits

Type of operator	Value of γ	Value of α	
		conj.	disj.
Dombi	0	$0<\alpha$	$\alpha<0$
Product	1	1	-1
Einstein	2	1	-1
Hamacher	$\gamma\in(0,\infty)$	1	-1
Drastic	∞	$0<\alpha$	$\alpha<0$
Min-max	0	∞	$-\infty$

7) introduced new forms of the Hamacher operators

$$o_H^{(\alpha)}(\mathbf{x}) = \frac{1}{1+\left(\frac{1}{\gamma_d}\left(\prod_{i=1}^n\left(1+\gamma_d\left(\frac{1-x_i}{x_i}\right)^\alpha\right)-1\right)\right)^{1/\alpha}}$$

8) introduced new forms of the Einstein operators

$$o_{GD,2}^{(\alpha)}(\mathbf{x}) = \frac{1}{1+2\left(\prod_{i=1}^n\left(1+2\left(\frac{1-x_i}{x_i}\right)^\alpha\right)-1\right)^{1/\alpha}}$$

9) shown that the addition of several velocities in the framework of special relativity is:

$$v = \frac{c}{1+2\left(\prod_{i=1}^n\left(1+2\frac{v_i}{c-v_i}\right)-1\right)^{-1}}.$$

References

[1] L. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] H. Hamacher, *Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien*. Rita G. Fischer Verlag, Frankfurt, 1978.

[3] A. Kuwagaki, “Sur l’équation fonctionnelle rationnelle de la fonction inconnue de deux variables,” *Memoirs of the College of Science, Vol. XXVIII, Mathematics No. 2*, 1952.

[4] K. Menger, “Statistical metric spaces,” *Proc. Nat. Acad. Sci. USA*, vol. 28, pp. 535–537, 1942.

[5] B. Schweizer and A. Sklar, “Espaces métiques aléatoires,” *C. R. Acad. Sci. Paris. Sér. A*, vol. 247, pp. 2092–2094, 1958.

[6] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*. Kluwer, 2000.

[7] R.L. Keeney, H. Raiffa, *Decision with multiple objectives: preference and value tradeoffs*. John Wiley and Sons, 1976.

[8] J. Aczél, *Lectures on functional equations and their applications*. Academic Press, New York, 1966.

[9] J. Dombi, “A general class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators,” *Fuzzy Sets and Systems*, vol. 8, pp. 149–163, 1982.

[10] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*. North-Holland, Amsterdam, 1983.

[11] M. Sugeno, “Fuzzy measures and fuzzy integrals: a survey,” in *Fuzzy Automata and Decision Processes*. North-Holland, Amsterdam, 1977, pp. 89–102.