Ranking alternatives with preference function based on the pliant concept

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***Abstract* – A novel preference function is introduced for multi-criteria decision management. The function is based on the pliant concept and the alternative choices are modelled with pliant numbers, i.e. softened arithmetic inequalities. Linear and sigmoid pliant numbers are considered. Its shown that the preference function fulfills shift invariance, transitivity and monotonicity properties.**

***Keywords* – Dombi operator, multi-criteria decision management, preference relation.**

I. INTRODUCTION

Multi-criteria decision management applications are important tools for decision makers. One of the difficulties in decision management is the comparison of alternative choices. The alternatives are often described with human words or given with fuzzy boundaries and as such cannot be ranked easily. Fuzzy theory [1] provides a mathematical foundation for modelling imprecise values and elusive human words [2]. There has already been a lot of effort to aid the decision process with fuzzy ranking algorithms.

One of the earliest fuzzy ranking function comes from Shimura and is based on a comparison method which is adopted for psychological test [3]. Buckley and Chanas [4],[5],[6] provided a fast ranking method using interval analysis. The ranking algorithm introduced by Cheng [7] is based on calculating distances. Delgado et al. [8] gave an ordering procedure using fuzzy relations and fuzzy measures. The preference relation described by Kundu [9] utilizes a fuzzy leftness relation on intervals.

Despite the many different approaches, there is still no consensus which ranking method is the best suitable for applications. The problem arises from the requirements that the ranking algorithm should run fast and have all the properties that a ranking procedure over crisply defined objects has.

In this paper we propose a novel preference ranking method based on the pliant concept [10]. Pliant ranking provides a pairwise choice between two alternatives and can be calculated easily. Our algorithm models the various alternatives with pliant numbers and defines a preference relation over them. We prove that pliant ranking fulfills all the necessary properties associated with preference methods, i.e. invariance, transitivity and monotonicity.

II. BASIC DEFINITIONS

The pliant ranking method is based on the pliant concept. We start by introducing the most important definitions of pliant systems.

Pliant systems consist of an operator family from the nilpotent and the strict operator class, defined as follows. Let $f_c(x)$ be a generator function of conjunction and let $f_d(x)$ be a generator function of disjunction. The *additive pliant system* is defined by the $f_c(x) + f_d(x) = 1$ equation. The Łukasiewicz operator family is an additive pliant system. The *multiplicative pliant system* is defined by the $f_c(x) \cdot f_d(x) = 1$ equation. The Dombi operator family [11] is a multiplicative pliant system.

A. Pliant Inequalities

Fuzzy theory can be divided into fuzzy logic and fuzzy arithmetics. Fuzzy logic deals with logical and set operations, while fuzzy arithmetics investigates fuzzy arithmetic operations and functions. The fuzzy membership function in both cases are heavily discussed and different approaches were developed to describe imprecise perceptions such as the logical value *young* and the arithmetical quantity *approximately* 3. The ”membership” functions in pliant systems are based on inequalities, e.g. young could be described as less than 22 years old, and approximately 3 as greater than 2 but less than 4. Fuzzy logical and arithmetical values can be uniformly described using *pliant inequalities*. Pliant inequalities are created by *softening* the arithmetic inequalities, i.e. replacing the characteristic function with a monotone, threshold like function. Thus an arithmetic inequality $a < x$ becomes a pliant inequality a monotone increasing function on the extended real line with values in $[0, 1]$. The pliant inequality reaches its threshold value (usually $\frac{1}{2}$) at point a and a λ variable controls the softness of the inequality.

In pliant systems the generator function of the aggregation operator induces the pliant inequality. In case of the Łukasiewicz operator family it is the linear function defined in the following manner.

Definition II.1: An additive pliant inequality is a linear function $l(x)$,

$$l(x) = \lambda(x - a) + \frac{1}{2}, \tag{1}$$

where a is the mean value and λ is the tangent of the linear function as shown in Fig. 1.

The inverse of $l(x)$ is

$$l^{-1}(y) = \frac{y - \frac{1}{2}}{\lambda} + a. \tag{2}$$

A linear function given by its mean value is a simple way to soften an arithmetic inequality.

Notation II.2: The additive pliant inequality is denoted as

$$[a <_\lambda x] = l(x) = \lambda(x - a) + \frac{1}{2}. \tag{3}$$

In case of the Dombi operator family the aggregation operator induces the sigmoid function.

Definition II.3: A multiplicative pliant inequality is a sigmoid function,

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}}, \tag{4}$$

where a is the mean value and λ is a softness parameter.

The inverse function of $\sigma_a^{(\lambda)}(x)$ is

$$\left(\sigma_a^{(\lambda)}\right)^{-1}(y) = -\frac{1}{\lambda}\ln\left(\frac{1-y}{y}\right) + a. \tag{5}$$

Notation II.4: The multiplicative pliant inequality is denoted as

$$\{a <_\lambda x\} = \sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}}. \tag{6}$$

B. Pliant arithmetics

Arithmetics in pliant systems is based on the characterization of imprecise quantities with $a < x$ and $x < b$ bounding inequalities. Pliant numbers are created by softening the inequalities i.e., replacing the crisp characteristic function with pliant inequalities and applying a pliant conjunction operator. A pliant number constructed with e.g. sigmoid pliant inequalities can be denoted as $\{a <_{\lambda_1} x\} \wedge \{x <_{\lambda_2} b\} = \{a <_{\lambda_1} x <_{\lambda_2} b\}$. The pliant inequalities $\{a <_{\lambda_1} x\}$, $\{x <_{\lambda_2} b\}$ are referred as the left side and right side of the pliant number respectively. The construction of pliant numbers differs from the classical fuzzy number representation, it arises from the arithmetic inequalities. Arithmetic operations are carried out by separately applying them to the left sides and to the right sides of pliant numbers. This decomposition allows us to treat pliant numbers as monotone functions when carrying out pliant arithmetic operations.

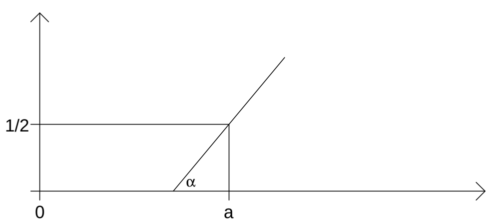

Fig. 1. Linear function given by its mean value a and $\lambda = \tan\alpha$.

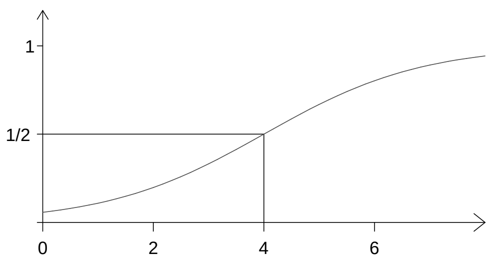

Fig. 2. Sigmoid with $\lambda = 0.7$ and $a = 4$ parameters.

C. Preference relations

In the following the fuzzy preference relation over fuzzy numbers and some important properties are defined. Fuzzy numbers are denoted with capital letters (A, B, ...) or with the corresponding membership functions ($\mu_A(x)$, $\mu_B(x)$, ...). The set of fuzzy numbers is denoted with $\mathbb{F}$. The only assumption about the membership functions made here is that they are continuous. This allow that the definitions can also be applied to pliant numbers introduced in the previous section.

Definition II.5: The function $P(X, Y)$ is a fuzzy preference relation, if P is continuous, monotone and maps from $\mathbb{F} \times \mathbb{F}$ to the $[0, 1]$ interval. For any $X \in \mathbb{F}$, $P(X, X) = \nu$, where constant ν is called the *threshold value* of the fuzzy preference P.

Let $A, B \in \mathbb{F}$ be arbitrary fuzzy numbers and $P(X, Y)$ a fuzzy preference relation. We say that A is better than B and denote it as

$$A >_P B \qquad \text{iff.} \qquad P(A, B) > \nu. \tag{7}$$

The value of the preference $P(A, B)$ measures how much alternative A is better than alternative B.

The following properties are considered to be important for fuzzy preference relations.

Definition II.6: (Shift invariance) Let $P(X, Y)$ be a fuzzy preference relation, $A, B \in \mathbb{F}$ and $d \in \mathbb{R}$.

Let $\hat{A}$ and $\hat{B}$ be defined as $\mu_{\hat{A}}(x) = \mu_A(x-d)$ and $\mu_{\hat{B}}(x) = \mu_B(x-d)$, i.e. by shifting the fuzzy numbers A and B along the horizontal axis with d. The relation $P(X, Y)$ is called shift invariant if $P(A, B) = P(\hat{A}, \hat{B})$.

Definition II.7: (Transitivity) Let $P(X, Y)$ be a fuzzy preference relation, $A, B, C \in \mathbb{F}$. The relation $P(X, Y)$ is transitive if

$$P(A, B) \geq \nu \quad \text{and} \quad P(B, C) \geq \nu, \quad \text{then} \quad P(A, C) \geq \nu. \tag{8}$$

Definition II.8: (Shift monotonicity) Let $P(X, Y)$ be a fuzzy preference relation, $A, B \in \mathbb{F}$ fuzzy numbers, and let $\hat{A}$ defined as $\mu_{\hat{A}}(x) = \mu_A(x - d)$.

The relation $P(X, Y)$ is called shift monotone iff.

$$\forall d > 0 \in \mathbb{R} : P(\hat{A}, B) > P(A, B) \tag{9}$$

$$\forall d < 0 \in \mathbb{R} : P(\hat{A}, B) < P(A, B). \tag{10}$$

III. OVERVIEW

Piant ranking defines a preference relation to provide pairwise comparison of pliant numbers. First, the draft pliant preference algorithm is presented. The calculation procedure is based on the idea used in pliant arithmetic operations. The algorithm is then applied to the class of additive and multiplicative pliant numbers and it is shown that in both cases the preference relation has all the required properties.

Pliant Preference Algorithm III.1: Let A, B be pliant numbers composed of strictly monotone pliant inequalities. The pliant preference $P(A,B)$, meaning how much A is greater than B is calculated as follows.

1. Decompose A and B to their left and right hand side pliant inequalities denoted as A_l, B_l and A_r, B_r respectively.
2. Calculate the inverse of the pliant inequalities. Let the inverse functions be denoted as A_l^{-1}, B_l^{-1} and A_r^{-1}, B_r^{-1}.
3. Calculate the left hand side preference function $p_l(A_l^{-1}, B_l^{-1})(x)$ with

$$p_l(A_l^{-1}, B_l^{-1})(x) = \frac{1}{1 + e^{-\lambda(A_l^{-1}(x) - B_l^{-1}(x))}}, \quad (11)$$

where $\lambda \in \mathbb{R}$ defines the *sharpness* of comparison. Calculation of the right hand side preference function $p_r(A_r^{-1}, B_r^{-1})(x)$ is carried out analogously.

4. Aggregate the left and right hand side preference functions with the aggregation function

$$a(p_l, p_r)(x) = \frac{1}{1 + \left(\frac{1-p_l(x)}{p_l(x)}\right)\left(\frac{1-p_r(x)}{p_r(x)}\right)} \quad (12)$$

5. The pliant preference relation $P(A,B)$ is then calculated by integrating the aggregated preference function over the unit interval,

$$P(A,B) = \int_0^1 a(p_l, p_r)(x)dx. \quad (13)$$

The algorithm starts with the calculation of a separate preference function for the left and right hand side of the pliant numbers. The pliant numbers are decomposed to pliant inequalities and their inverse functions are calculated. The inverse functions allow us to measure the difference of the functions by subtraction and normalize it to the unit interval by using a sigmoid function. The λ parameter controls the sharpness of the sigmoid function. It can be used to control the margin of error and by letting $\lambda \to \infty$ the preference function can be turned into crisp preference, i.e. returning only 0 or 1. The left and right hand side preference functions are combined using an aggregation function and the pliant preference value is calculated by integrating the function over the unit interval.

IV. PLIANT PREFERENCE PROPERTIES

Lemma IV.1: The threshold value of the pliant preference given in Algorithm III.1 is $\frac{1}{2}$.

Proof: Let A be a pliant number composed of A_l and A_r strictly monotone pliant inequalities. Let A_l^{-1} and A_r^{-1} denote the inverse of the pliant inequalities. Let us substitute the inverse functions directly into the pliant preference:

$$P(A,A) = \int_0^1 \frac{1}{1 + e^{-\lambda(A_l^{-1} - A_l^{-1} + A_r^{-1} - A_r^{-1})(x)}} dx \quad (14)$$

$$= \int_0^1 \frac{1}{1+e^0} dx = \frac{1}{2}. \quad (15)$$

■

Lemma IV.2: The pliant preference given in Algorithm III.1 is shift invariant.

Proof: Let $d \in \mathbb{R}$, let $\hat{A}$ and $\hat{B}$ be defined as $\mu_{\hat{A}}(x) = \mu_A(x-d)$ and $\mu_{\hat{B}}(x) = \mu_B(x-d)$. This implies that $\hat{A}_l(x) = A_l(x-d)$ and $\hat{B}_l(x) = B_l(x-d)$. For the inverse functions it is easy to see that $\hat{A}_l^{-1}(x) = A_l^{-1}(x) + d$ and $\hat{B}_l^{-1}(x) = B_l^{-1}(x) + d$. From here

$$p_l(\hat{A}_l^{-1}, \hat{B}_l^{-1})(x) = \frac{1}{1 + e^{-(A_l^{-1}(x) + d - B_l^{-1}(x) - d)}} \quad (16)$$

$$= p_l(A_l^{-1}, B_l^{-1})(x). \quad (17)$$

Therefore the left pliant preference function is independent of shifting the pliant numbers by d. The same holds for the right pliant preference function, thus the pliant preference is shift invariant. ■

The rest of the properties are examined separately in the case of additive and multiplicative pliant numbers.

V. ADDITIVE PLIANT PROPERTIES

This section examines the pliant ranking of additive pliant numbers, i.e. pliant numbers constructed from additive pliant inequalities. This case the λ parameter of the pliant preference function is set to 1. Let A, B be additive pliant numbers. The aggregated preference function can be written as:

$$a(p_l, p_r)(x) = \frac{1}{1 + e^{-(A_l^{-1}(x) - B_l^{-1}(x) + A_r^{-1}(x) - B_r^{-1}(x))}}. \quad (18)$$

Let the inverse functions be given as

$$A_l^{-1}(x) = \frac{x - \frac{1}{2}}{\lambda_l^A} + a_l^A, \quad A_r^{-1}(x) = \frac{x - \frac{1}{2}}{\lambda_r^A} + a_r^A \quad (19)$$

$$B_l^{-1}(x) = \frac{x - \frac{1}{2}}{\lambda_l^B} + a_l^B, \quad B_r^{-1}(x) = \frac{x - \frac{1}{2}}{\lambda_r^B} + a_r^B. \quad (20)$$

The integral can be written in a simplified form

$$P(A,B) = \int_0^1 a(p_l, p_r)(x)dx \quad (21)$$

$$= \int_0^1 \frac{1}{1 + e^{-(Vx+W)}} dx, \quad (22)$$

where

$$V = \frac{1}{\lambda_l^A} - \frac{1}{\lambda_l^B} + \frac{1}{\lambda_r^A} - \frac{1}{\lambda_r^B}, \quad (23)$$

$$W = -\frac{1}{2}V + a_l^A - a_l^B + a_r^A - a_r^B. \quad (24)$$

Calculation of the integral gives

$$P(A,B) = \left[-\frac{\ln(e^{-(Vx+W)})}{V} + \frac{\ln(1+e^{-(Vx+W)})}{V} \right]_0^1 \quad (25)$$

$$= \frac{\ln(e^{(V+W)}+1)}{V} - \frac{\ln(e^W+1)}{V}. \quad (26)$$

Thus the resulting integral can be calculated exactly and gives a value between $[0,1]$, which can be directly used in multi-valued logic.

Lemma V.1: The pliant preference given in Algorithm III.1 is transitive.

Proof: Let A, B, C additive pliant numbers with strictly monotone pliant inequalities and $P(A,B) \geq \nu$ and $P(B,C) \geq \nu$ hold. We show that $P(A,C) \geq \nu$.

From Lemma IV.1 we know that $\nu = \frac{1}{2}$. Using Equation 18 the two inequalities can be transformed to

$$-(A_l^{-1}(x) - B_l^{-1}(x) + A_r^{-1}(x) - B_r^{-1}(x)) \geq 0 \quad (27)$$

$$-(B_l^{-1}(x) - C_l^{-1}(x) + B_r^{-1}(x) - C_r^{-1}(x)) \geq 0 \quad (28)$$

By summing up the inequalities and transforming back the result using $ln1 = 0$, we get the required inequality

$$\frac{1}{1+e^{-(A_l^{-1}(x)-C_l^{-1}(x)+A_r^{-1}(x)-C_r^{-1}(x))}} \geq \frac{1}{2}. \quad (29)$$

■

Lemma V.2: The pliant preference given in Algorithm III.1 is shift monotone.

Proof: Let A, B additive pliant numbers with strictly monotone pliant inequalities $d \in \mathbb{R}$ and let $\hat{A}$ defined as $\mu_{\hat{A}}(x) = \mu_A(x-d)$.

Using the notation of Equation 23 and 24, it is easy to check that

$$\hat{V} = V \quad \text{and} \quad \hat{W} = W + 2d. \quad (30)$$

Thus the preference $P(\hat{A}, B)$ is

$$P(\hat{A},B) = \frac{\ln(e^{(V+W+2d)}+1)}{V} - \frac{\ln(e^{(W+2d)}+1)}{V}. \quad (31)$$

Calculating the change of the preference value we have

$$\left(P(\hat{A},B) - P(A,B)\right)(d) = \quad (32)$$

$$= \left(\frac{\ln(e^{(V+W+2d)}+1)}{V} - \frac{\ln(e^{(V+W)}+1)}{V} \right) \quad (33)$$

$$- \left(\frac{\ln(e^{(W+2d)}+1)}{V} - \frac{\ln(e^W+1)}{V} \right). \quad (34)$$

The function is monotone, continuous and therefore the preference function fulfills the monotonicity properties. ■

VI. MULTIPLICATIVE PLIANT PROPERTIES

This section examines the pliant ranking of multiplicative pliant numbers, i.e. pliant numbers constructed from sigmoid functions. Sigmoid functions appear in many natural processes thus their investigation in decision making is important. Let A, B be multiplicative pliant numbers. The aggregated preference function can be written as:

$$a(p_l,p_r)(x) = \frac{1}{1+e^{-\lambda(A_l^{-1}(x)-B_l^{-1}(x)+A_r^{-1}(x)-B_r^{-1}(x))}}. \quad (35)$$

Let the inverse functions be given as

$$A_l^{-1}(x) = -\frac{1}{\lambda_l^A}\ln\left(\frac{1-x}{x}\right) + a_l^A \quad (36)$$

$$A_r^{-1}(x) = -\frac{1}{\lambda_r^A}\ln\left(\frac{1-x}{x}\right) + a_r^A, \quad (37)$$

$$B_l^{-1}(x) = -\frac{1}{\lambda_l^B}\ln\left(\frac{1-x}{x}\right) + a_l^B \quad (38)$$

$$B_r^{-1}(x) = -\frac{1}{\lambda_r^B}\ln\left(\frac{1-x}{x}\right) + a_r^B. \quad (39)$$

The integral can be written in the form

$$P(A,B) = \int_0^1 a(p_l,p_r)(x)dx \quad (40)$$

$$= \int_0^1 \frac{1}{1+\left(\frac{1-x}{x}\right)^{-\lambda V} e^{-\lambda W}} dx, \quad (41)$$

where

$$V = -\frac{1}{\lambda_l^A} + \frac{1}{\lambda_l^B} - \frac{1}{\lambda_r^A} + \frac{1}{\lambda_r^B}, \quad (42)$$

$$W = a_l^A - a_l^B + a_r^A - a_r^B. \quad (43)$$

Unfortunately, the integral does not have a primitive function. The value of the preference function is approximated in the following manner. Let $\lambda' = -\lambda V$ then

$$P(A,B) = \int_0^1 \frac{1}{1+(\frac{\nu}{1-\nu})^{\lambda'}(\frac{x}{1-x})^{\lambda'}} dx \approx 1-\nu, \quad (44)$$

if λ' is large enough ($\lambda' \geq 1$), where

$$\nu = \frac{1}{1+e^{\frac{W}{V}}}. \quad (45)$$

The approximation is based on the relations

$$a(p_l,p_r) = \frac{1}{2} \quad \text{iff.} \quad x = \nu \quad \text{and} \quad (46)$$

$$\lim_{\lambda'\to\infty} a(p_l,p_r)^{-1} = \nu \quad \text{which gives} \quad (47)$$

$$\lim_{\lambda'\to\infty} P(A,B) = 1-\nu. \quad (48)$$

Lemma VI.1: The approxmiation of the pliant preference given in Algorithm III.1 is transitive and shift monotone.

Proof: The calculations are analogous to the additive case. ■

VII. CONCLUSION

In this paper a novel ranking method that can be used in multi-criteria decision management is introduced. Pliant ranking is based on the pliant preference function. The pliant preference algorithm provides a generic procedure for comparing pliant numbers with various membership functions. It was shown that in the case of linear and sigmoid membership function the pliant preference fulfills the shift invariance, transitivity and shift monotonicity properties.

REFERENCES

[1] L.A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, pp. 338–353, 1965.

[2] L.A. Zadeh, "A computational approach to linguistic quantifiers in natural language," *Comp. Math. Appl.*, vol. 9, pp. 149–184, 1983.

[3] M. Shimura, "Fuzzy sets concept in rank-ordering objects," *Journal of Mathematical Analysis and Applications*, vol. 43, pp. 717–733, 1973.

[4] J.J. Buckley, "Ranking alternatives using fuzzy numbers," *Fuzzy Sets and Systems*, vol. 15, pp. 21–31, 1985.

[5] J.J. Buckley, "Fuzzy hierarchical analysis," *Fuzzy Sets and Systems*, vol. 17, pp. 233–247, 1985.

[6] J.J. Buckley and S. Chanas, "A fast method of ranking alternatives using fuzzy numbers," *Fuzzy Sets and Systems*, vol. 30, pp. 337–339, 1989.

[7] C-H. Cheng, "A new approach for ranking fuzzy numbers by distance method," *Fuzzy Sets and Systems*, vol. 95, pp. 307–317, 1998.

[8] M. Delgado, J.L. Verdegay, and M.A. Vila, "A procedure for ranking fuzzy numbers using fuzzy relations," *Fuzzy Sets and Systems*, vol. 26, pp. 49–62, 1988.

[9] S. Kundu, "Preference relation on fuzzy utilities based on fuzzy leftness relation on intervals," *Fuzzy Sets and Systems*, vol. 97, pp. 183–191, 1998.

[10] J. Dombi, "Fuzzy arithmetic operations based on inequalities: The pliant concept," *Paper submitted to Fuzzy Sets and Systems*, 2005.

[11] J. Dombi, "General class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators," *Fuzzy Sets and Systems*, vol. 8, pp. 149–168, 1982.