Theoretical Concept of Modifiers and Hedges

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Abstract

Hedges and modifiers play an important role in fuzzy theory. While other areas as intersection, union, complement, aggregation, preference relation etc. are theoretically deeply studied, the hedges preserved its heuristic nature, and only some characteristics were studied - among other researchers - by Mattila. In our paper we give a general form of the hedges based on connectives building DeMorgan classes. It is an important corollary, that hedges are operator dependent. We proved that our proposals fulfill the most important features of hedges that can be expected.

1 Introduction

The concept of hedges and modifiers appears at the very beginning of fuzzy set theory. It is connected with an attempt to model meaning like ”very”, ”more or less”, ”somewhat”, ”rather” or ”quite”. The hedge modifies the shape of the fuzzy set, causing a change in the membership function. Thus a hedge transforms one fuzzy set into another new fuzzy set.

The modifier modifies the truth values of fuzzy logical statement. Hedges work on fuzzy sets and modifiers work on logic. We can handle both of them together if we transform the domain of the fuzzy set into the unit interval, and we choose the range of fuzzy sets (i.e. the membership functions) as usual to be in the unit interval.

In fuzzy reasoning systems, there are several different classes of hedge operators. There are hedges that intensify the characteristics of fuzzy set (very, extremely), that dilute the membership curve (somewhat, rather, quite) or that form the complement of a set (not). There are other types that approximate fuzzy region (about, near, close). In our work we don’t deal with this type of hedges. Hedges can be combined with each other (not very high).

The mechanics underlying hedge operators are generally heuristic by nature. This means that the membership transformations are not based on the mathematical theory of fuzzy operations. Zadeh’s original definition of the hedge ”very” (which intensifies the fuzzy membership) is

$$\tau(x) = x^2$$

where $x = \mu(t)$ is the membership function. Zadeh's definition of the hedge "somewhat" or "rather" (which dilutes the fuzzy membership) is

$$\tau(x) = x^{1/2}$$

where $x = \mu(t)$ is the membership function.

Why do we square the membership function or take square root for the "very" or the "somewhat" hedge? Should we use the same transformation even if we are using different operations for conjunction i.e. $\min(x, y)$ or $\max(0, x + y - 1)$? Why don't we use the cube of the membership value or take some intermediate value as 1.5?. In fact the hedge

$$\tau(x) = x^{1.3}$$

have been used frequently to implement the hedge "slightly". Because hedges are heuristic by nature, several well-known academic expert rejected the concept of hedges, as Cox states in [3].

As we mentioned before, modifiers are used in the logic part of fuzzy systems. The structure and the properties of modifiers are deeply studied by Mattila. The main purpose of his work was to create a connection between classical logic and some special topics of non-classical logic. He was looking for the structure of modifier operators and applied these to modal operators of alethic modal logic and to the logic of hedges. The work of Mattila is theoretical and does not give calculation procedures for the modifiers.

Our approach is on one hand practical (i.e. gives a calculation procedure), on the other hand theoretical, and operator dependent i.e. it orders different modifiers to different classes of connectives.

2 Basic Definitions

We start with the definition of the substantiating, and the weakening modifier. These modifiers are compositional modifiers.

Definition 1. *We say $\tau : [0, 1] \to [0, 1]$ is a substantiating modifier if*

$$\tau(x) < x \qquad \forall x \in [0, 1], \tag{1}$$

and a weakening modifier if

$$\tau(x) > x \qquad \forall x \in [0, 1]. \tag{2}$$

Definition 2. $n(x)$ *is a negation if*

1. $n : [0, 1] \to [0, 1]$*, continuous,*
2. *strictly decreasing,*
3. $n(0) = 1$*,* $n(1) = 0$*,*

4. *involutive:* $n(n(x)) = x$.

Using the negation we can associate a given modifier its dual modifier according to the following definition.

Definition 3. *Let τ and τ_* be modifiers. We say τ_* is the dual modifier associated with τ if for*

$$\tau_*(x) = n(\tau(n(x))) \tag{3}$$

where n is a negation.

It is easy to see if τ is a substantiating modifier then its dual τ_* is a weakening modifier and vice versa.

3 Modifiers Generated by DeMorgan classes

We can generate modifiers of different substantiating or weakening grades using conjunctive or disjunctive operators building DeMorgan classes [1]. This means that the base element is just one t-norm. It generates a t-conorm as its dual. Depending on the number of arguments of the operators we get different substantiating operators. The more the arguments the more substantiating the resulting modifier. Correspondingly, we use the same procedure for weakening modifiers using the t-conorm. We consider here only DeMorgan classes (c, d, n) where $c(x, y)$ is a conjunctive operator, $d(x, y)$ is a disjunctive operator and $n(x)$ is a negation, i.e.

$$d(x, y) = n(c(n(x), n(y))) \tag{4}$$

We introduce the modifiers on two basic classes of operators: for the strict monotonously increasing operators and for the nilpotent class.

The strict monotonously increasing operators have the following form:

$$c(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) \tag{5}$$

where f is the generator function of the operator. [4]

For example if $f(x) = \ln(x)$ then the strict monotonous operator is

$$c_{S_1}(x, y) = e^{\ln(x)+\ln(y)} = xy. \tag{6}$$

The other class is the class of bound or nilpotent operators. The representation theorem for this type of operator is

$$c(x_1, \ldots, x_n) = f^{-1}\left[\sum_{i=1}^{n} f(x_i)\right] \tag{7}$$

where f is the generator function of the operator [5] and the meaning of $[x]$ is

$$[x] = \begin{cases} (0) & \text{if} \quad x < 0 \\ (x) & \text{if} \quad 0 \leq x \leq 1 \\ (1) & \text{if} \quad x > 1 \end{cases} \tag{8}$$

and $f : [0,1] \to [0,1]$ is a continuous, strict monotonously decreasing function.

Let the negation be $1 - x$, then

$$d_S(x,y) = 1 - (1-x)(1-y) = x + y - xy. \tag{9}$$

If $f(x) = 1 - x$ then the bounded operator is

$$c_B(x,y) = [x + y - 1] = \max(0, x + y - 1), \tag{10}$$

$$d_B(x,y) = [x + y] = \min(1, x + y). \tag{11}$$

Definition 4. *The substantiating operator of grade m induced by the conjunctive operator is*

$$\tau^{(m)}(x) = c(\underbrace{x, \ldots, x}_{m}), \tag{12}$$

and the corresponding dual modifier i.e. the weakening modifier is

$$\tau_*^{(m)}(x) = d(\underbrace{x, \ldots, x}_{m}). \tag{13}$$

Using the representation theorem (equation 5 and 7) we get:

$$\tau_S^{(m)}(x) = f^{-1}\left(m \cdot f(x)\right), \tau_B^{(m)}(x) = f^{-1}\left[m \cdot f(x)\right]. \tag{14}$$

Because c, d and n build a DeMorgan class

$$\tau_*^{(m)}(x) = n\left(\tau^{(m)}(n(x))\right). \tag{15}$$

If $m = 1$ then

$$\tau^{(1)}(x) = \tau_*^{(1)}(x) = x \tag{16}$$

i.e. $\tau^{(1)}(x)$ is the identity modifier.

As a special case of the strict monotonous case we get

$$\tau_P^{(m)}(x) = x^m \tag{17}$$

$$\tau_{P^*}^{(m)}(x) = 1 - (1-x)^m \tag{18}$$

We call this modifier system the product modifier system.

Another modifier system is the linear bound modifier system:

$$\tau_L^{(m)}(x) = [mx - (m-1)] \tag{19}$$

$$\tau_{L^*}^{(m)}(x) = [mx] \tag{20}$$

4 Logic and Modifiers

Mattila in his thesis [2] gave axiomatical characterization of modifiers. From the axioms he derived important properties which are consistent with the classical logic and are close to the modal logic. We show that our construction also fulfills these properties.

Proposition 1. *It is valid*

$$\tau_*(c(x,y)) = c(\tau(x),\tau(y))) \tag{21}$$
$$\tau_*(d(x,y)) = d(\tau_*(x),\tau_*(y))) \tag{22}$$

For simplicity we left the grade of modifiers. The proposition is true for monotone and bounded logic as well.

Proof.

$$\begin{aligned}\tau(c(x,y)) &= f^{-1}(n\cdot c(x,y)) = f^{-1}(n\cdot f(f^{-1}(f(x)+f(y)))) = f^{-1}(n\cdot f(x)+n\cdot f(y))\\ &= f^{-1}(f(f^{-1}(n\cdot f(x)))+f(f^{-1}(n\cdot f(y)))) = f^{-1}(f(\tau(x))+f(\tau(y)))\\ &= c(\tau(x),\tau(y)).\end{aligned}$$

The proposition for the disjunctive operator can be proved in a similar way. The bounded case needs only discussion on the bounds. □

Proposition 2.

$$n(\tau_*(d(x,y))) = c(n(\tau_*(x)),n(\tau_*(y))) \tag{23}$$
$$n(\tau(c(x,y))) = d(n(\tau(x)),n(\tau(y))) \tag{24}$$

Proof. Let $x := n(x)$, then

$$n(\tau_*(d(n(x),n(y)))) = c(\underbrace{n(\tau_*(n}_{\tau}(x))),\underbrace{n(\tau_*(n}_{\tau}(y)))) = c(\tau(x),\tau(y))$$

We can write the left side in the following way:

$$n(\tau_*(n(\underbrace{n(d(n(x),n(y)))}_{c(x,y)}))) = \underbrace{n(\tau_*(n}_{\tau}(c(x,y)))) = \tau(c(x,y)).$$

So we get

$$\tau(c(x,y)) = c(\tau(x),\tau(y))$$

the result of proposition 1. The second equation can be proved in a similar way. □

Proposition 3.

$$c(\tau^{(m)}(x),\tau^{(n)}(x)) = \tau^{(m+n)}(x) \tag{25}$$
$$d(\tau_*^{(m)}(x),\tau_*^{(n)}(x)) = \tau_*^{(m+n)}(x) \tag{26}$$

Proof. We get results by using the representation theorem of $c(x, y)$ and $d(x, y)$ and the definition of $\tau(x)$. □

Proposition 4.

$$\tau^{(m)}(\tau^{(n)}(x)) = \tau^{(m \cdot n)}(x) \tag{27}$$

$$\tau_*^{(m)}(\tau_*^{(n)}(x)) = \tau_*^{(m \cdot n)}(x) \tag{28}$$

Proof.

$$\tau^{(m)}(\tau^{(n)}(x)) = \tau^{(m)}(f^{-1}(n \cdot f(x))) = f^{-1}(m \cdot f(f^{-1}(n \cdot f(x)))) =$$
$$= f^{-1}(m \cdot n \cdot f(x)) = \tau^{(m \cdot n)}(x).$$

□

5 Conclusion

We have given operator dependent representation of hedges, which is consistent with the theoretical result developed in classical logic.

References

[1] J. Dombi. A General Class of Fuzzy Operators, the DeMorgan Class of Fuzzy Operators and Fuzziness Measures Induced by Fuzzy Operators. *Fuzzy Sets and Systems* **8**, 1982

[2] J. Mattila. Calculus of Modifier Operators in Fuzzy Logic. Ph.D. Thesis, University of Turku, 1984.

[3] E. Cox. The Fuzzy Systems Handbook. AP Professional, 1999.

[4] Aczel J. Lectures on Functional Equations and Applications. Academic Press, New York.

[5] Ling, C.H. Representation of Associative Functions and Statistical Triangle Inequalities. Publ. Math. Debrecen **12**, 189-212, 1965