
Common Preference Model for Various Decision Models

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Summary. Recently many axiomatic results concerning modelling preferences and aggregation procedures have been developed of conjoint measurement and based on social choice theory. We propose a general tool which differs from the above mentioned approaches. The so called Bounded System is general and also helps to build new MCDM tools and the classical procedures (ELECTRE, PROMETHEE, TACTIC, MAUT, Lexicographic decision etc.) can be described as well with this. Logic (conjunction, disjunction, negation) and decision operators (mean, preference relation and unary operators) belong to the Bounded System. It seems to be a very powerful tool for the real world decision problems.

1 Introduction

Decision-makers use aggregation procedures, using value functions or preference relation. But the aggregation procedure can not be independent from the value function or from the preference relation. The conjoint measurement handle both not independently. The advantage of this model is that it gives a general framework modelling the decision situation, and this is the reason why we summarize the main results of the various types of conjoint measurement in section 2. But the conjoint measurement does not tell us much about how we should make the decision-making. Marchant [28] in his beautiful article also made the same conclusion. If a decision-maker is facing an aggregation problem he doesn't know which aggregation procedure is he going to use, MAUT or PROMETHEE II. To choose the MCDM method also seems a multicriteria decision problem. The conjoint measurement doesn't help us to choose one of them. Marchant in his article is interested in characterization of the aggregation procedure. He considered, that the social choice theory has relevance to solve this problem. He summarized all types of axioms and described the aggregation procedure depending on its different subsets. The result was many different aggregation procedures were described. Therefore characterization of these procedures will be comparable from the point of view of the axioms. The main problem is the assessment problem and that an even more general framework is needed. An other way to model a real world decision situation is to take into account the interaction between the criteria (Grabish [22]). The difficulties arise when we recognize that the assessment process is in this case increasing rapidly. Our Bounded System on

one hand is easy to understand and on the other hand the assessment of the parameters (weights, thresholds etc.) is also easy by using a learning algorithm, but because the resulting system is heterogenous it's hard to find a good characterization.

2 Conjoint Measurement

Conjoint measurement was first developed in psychology [27] and economics [6] in the beginning of the '60s. After its development people working in decision analysis realized that the conjoint measurement could also be used as a tool to preference structures.

The starting point of most works in decision theory is a binary relation \succsim as a set of objects A . This binary relation is interpreted as an “at least as good as” relation between alternatives.

Manipulating a binary relation can be quite difficult as soon as the set of objects is large. The traditional way of modelling the preferences of a decision-maker consist in assuming the existence of a value function u such that an alternative a is at least as good as an alternative b ($a \succsim b$) if and only if

$$u(a) \geq u(b) \quad (1)$$

for all $a, b \in A$. The function u is called the value function. When such numerical representation is possible we can use classical optimization techniques to find the most preferred elements in A . The main task of the analyst is therefore to assess u .

The condition (1) says that \succsim is complete and transitive. When A is finite or countable infinite, then these two conditions are not only necessary but also sufficient to build a value function satisfying (1)

In a multicriteria context each alternative a is seen as a vector (x_1, \dots, x_n) of evaluations on n attributes. In such a context the assessment of a value function u is not an easy task. It usually requires the specification of a particular form for u . The additive decomposition of traditional conjoint measurement models are in which:

$$a \succsim b \iff \sum_{i=1}^n u_i(x_i) \geq \sum_{i=1}^n u_i(y_i) \quad (2)$$

where the u_i are real valued functions on the sets of the i -th attribute, that $a = (x_1, x_2, \dots, x_n)$ and $b = (y_1, y_2, \dots, y_n)$.

Many techniques have been proposed ([12], [14], [20], [24]) to assess the “partial value functions” u_i for modelling preferences. Several axiom systems have been developed, but even in finite case the axiom system is hardly interpretable and testable.

The traditional model of conjoint measurement is not always appropriate for modelling preferences because the indifference (the symmetric part of \succsim)

may not be transitive, the preference relation may not be a complete relation (i.e. some alternatives may be incompatible) and the relation between the criterions may be more complex than with the additive model.

Several extensions of model (2) have been proposed. The first extension consists of replacing the additivity requirement by a general function. Krantz [25] introduced the following model

$$a \succsim b \iff F(u_1(x_1), \dots, u_n(x_n)) \geq F(u_1(y_1), \dots, u_n(y_n)) \quad (3)$$

where F is strictly increasing in all its arguments. Such a model allows interaction between attributes that are more complex than in an additive model. Necessary and sufficient conditions for (3) consist of a transitivity and completeness requirement together with a single cancellation condition. (3) is more general compared to (2) and can be characterized using a single axiom system.

(1), (2) and (3) imply that \succsim is complete and transitive. Many researchers have argued that the transitivity hypothesis is most unlikely. Tversky [33] was one of the first to propose such a model generalizing (2), known as the additive difference model in which

$$a \succsim b \iff \sum_{i=1}^n \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \quad (4)$$

where Φ_i are increasing and odd functions. It is clear that (4) allows for intransitivity \succsim but implies completeness. When attention is restricted to the comparison of objects that only differ on one attribute (4) as well as the other above mentioned models imply that the preference relation is independent from the remaining $n - 1$ attributes.

As suggested by Bouyssou [3] and Fishburn [15], [16] and Vind [34] the subtractivity requirement in (4) can be dropped. This leads to non transitive additive conjoint measurement models in which:

$$a \succsim b \iff \sum_{i=1}^n p_i(x_i, y_i) \geq 0 \quad (5)$$

where p_i are real valued functions. Model (5) is a generalization of model (3). Opposite to (3) it allows for intransitive and incomplete partial preferences [17] and several axiom systems have been proposed to characterize them. Model (5) involves the addition operation and the axiomatic analysis is very sophisticated.

The most general model introduced by Goldstein [21] is without transitivity and completeness and as a generalization of (5) dropping the additivity

$$a \succsim b \iff F(p_1(x_1, y_1), p_2(x_2, y_2), \dots, p_n(x_n, y_n)) \geq 0 \quad (6)$$

where F is a strictly increasing function and p_i is nondecreasing in its first argument and nonincreasing in the second for all i .

Model (6) is very general and shows fundamental features. The characterization of (6) is based on quaternary relation encoded the comparison of pairs of levels in each criterion (i.e. comparison of differences of preference). The model is fully discussed by Bouyssou and Pirlot [2], [4].

3 Bounded Logic

Our need for the multicriteria decision-making is a special continuously valued logic with some special requirements. Since Zadeh introduced fuzzy logic a wide variety of connectives were studied and from the theory of many valued logic reconsidered. For our purposes the Lukasiewicz logic fits well. This class of connectives have other names as t-norms with zero division or nilpotent class or bounded sums.

3.1 Negations

Negation is a unary operation that identify the value of *not P* for proposition *P*. We can use a valuation for a true proposition 1 and for the false proposition 0. We suppose that there exists a $n : [0, 1] \rightarrow [0, 1]$ strictly continuously decreasing function such that:

$$n(0) = 1 \quad \text{and} \quad n(1) = 0$$

and

$$n(n(x)) = x$$

which means that the negation is involutive. It is called strong negation. The representative theorem of strong negation was obtained by Trillas [32].

Theorem 1. *$n(x)$ is a strong negation iff it is representable in the following form:*

$$n(x) = f^{-1}(1 - f(x))$$

where $f(x)$ is the generator function of the negation, $f : [0, 1] \rightarrow [0, 1]$, continuous and strictly increasing function, $f(0) = 0$ and $f(1) = 1$.

The neutral value of the negation is defined by

$$n(\nu_*) = \nu_*$$

(This is called as the fix point or eigenvalue of the negation.) Using the representation theorem theory we get $2 \cdot n(\nu_*) = 1$ and

$$n_{\nu_*}(x) = f^{-1}(2 \cdot f(\nu_*) - f(x)).$$

3.2 Conjunction and Disjunction

Conjunction and disjunction are two placed functions with similar properties. Our requirements are the following:

$$c : [0, 1] \rightarrow [0, 1] \quad d : [0, 1] \rightarrow [0, 1]$$

1. c and d are continuous functions
2. c and d are nondecreasing in both arguments
3. c and d are compatible with the two valued logic

$$\begin{array}{cccc} c(0, 0) = 0 & c(0, 1) = 0 & c(1, 0) = 0 & c(1, 1) = 1 \\ d(0, 0) = 0 & d(0, 1) = 1 & d(1, 0) = 1 & d(1, 1) = 1 \end{array}$$

4. c and d are associative

$$\begin{array}{l} c(x, c(y, z)) = c(c(x, y), z) \\ d(x, d(y, z)) = d(d(x, y), z) \end{array}$$

5. c and d are Archimedean if

$$\begin{array}{ll} c(x, x) < x & x \in (0, 1) \\ d(x, x) > x & x \in (0, 1) \end{array}$$

In classical logic the law of contradiction and excluded middle play an important role. We will call both the classification property.

6. c and d have the classification property

$$\begin{array}{l} c(x, n(x)) = 0 \\ d(x, n(x)) = 1 \end{array}$$

We do not want to choose c , d and n independently. In classical logic the DeMorgan identity hold for the three connectives.

7. c , d and n build DeMorgan triples

$$n(c(x, y)) = d(n(x), n(y))$$

where $n(x)$ is the same as in property 6.

A logic with requirements 1-7 is called bounded logic. We state here the representation theorem of Ling [26]. In fact the theorem can be deduced from previously known results of Faucett [13] and Moster and Shields [29]. We are also using the results of Fodor and Roubens [19] and Dombi [9].

To avoid the introduction of the pseudoinverse of function f we are using the following notation.

$$[x] = \begin{cases} 1 & \text{if } x > 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \end{cases}$$

Theorem 2. c , d and n is bounded logic iff

$$\begin{aligned}d(x, y) &= f^{-1} [f(x) + f(y)] \\c(x, y) &= f^{-1} [f(x) + f(y) - 1] \\n(x) &= f^{-1} [1 - f(x)]\end{aligned}$$

where f is the generator function of the logic with $f : [0, 1] \rightarrow [0, 1]$ strictly monotonously increasing function and f is unique.

3.3 Implications

Several approaches exist to the definition of implications. We consider an implication as a two fold function, and we are interested in implication in bounded logic. Natural extension is to define $i(x, y)$ with *not* x or y , then

$$i(x, y) = f^{-1} [f(y) - f(x) + 1].$$

The implication can be defined in an other way:

$$i_R(x, y) = \sup_t (c(x, t) < y)$$

This is the R-implication. In bounded logic the two implications are the same, and $i(x, y)$ has several other good properties. It can also be characterized by axioms [19]. Without completeness we mention some of them:

1. Continuity

$$i : [0, 1] \rightarrow [0, 1]$$

2. Monotonicity

$$\begin{aligned}\text{if } x \leq x' & \quad \text{then } i(x', y) \leq i(x, y) \\ \text{if } y \leq y' & \quad \text{then } i(x, y) \geq i(x, y')\end{aligned}$$

3. Compatibility with the two valued logic

$$\begin{aligned}i(0, 0) &= 1 & i(1, 1) &= 1 \\ i(0, 1) &= 1 & i(1, 0) &= 0\end{aligned}$$

4. The principle of exchangeability

$$i(x, i(y, z)) = i(y, i(x, z))$$

5. The principle of contrapositive symmetry

$$i(x, y) = i(n(y), n(x))$$

6. The validity of boundary conditions

| | |
|-----------------------------|------------------|
| falsity principle | $i(0, x) = 1$ |
| neutrality principle | $i(1, x) = x$ |
| truth principle | $i(x, 1) = 1$ |
| compatibility with negation | $i(x, 0) = n(x)$ |

7. The identity principle

$$i(x, x) = 1$$

8. The generalized identity principle

$$i(x, y) = 1 \iff x \leq y$$

Property 4 plays a similar role as the associativity.

It is clear that the implication has very similar properties to the preference relation. Fodor and Roubens [19] suggested that this is the right way of modelling preferences in multicriteria decision modelling. Our view differs from theirs.

4 Bounded System

The Bounded System is an extension of the bounded logic with other operators as mean, preference relation and unary operators.

4.1 Mean

First we introduce the aggregation, which is the weighted form of the conjunctive and disjunctive operators of the bounded logic. We get the same form by using the representation theorem of the associative function equation from the conjunction and disjunction operators.

$$m(x, y) = f^{-1} \left(\frac{1}{2} (f(x) + f(y)) \right)$$

or generally

$$m^*(\mathbf{x}, \mathbf{w}) = f^{-1} \left(\sum_{i=1}^n w_i f(x_i) \right) \quad \sum_{i=1}^n w_i = 1 \quad w_i \geq 0$$

where f is the generator function of the operator, $f : [0, 1] \rightarrow [0, 1]$ continuous and strictly monotonously increasing, w_i are the weights (importance functions).

The mean operator is not logical because

$$m(0, 1) = f^{-1} \left(\frac{1}{2} \right)$$

and $f^{-1} \left(\frac{1}{2} \right) \neq 0$ and $f^{-1} \left(\frac{1}{2} \right) \neq 1$.

The mean operator of the bounded logic has several interesting properties. Let us summarize the most important ones:

1. Continuity

$$m : [0, 1] \times [0, 1] \rightarrow [0, 1]$$

2. Strictly monotonously increasing.
3. Compatibility

$$m(0, 0) = 0 \quad m(1, 1) = 1$$

4. Strict compensationness

$$\min(x_i) < m(\mathbf{x}, \mathbf{w}) < \max(x_i)$$

if not all x_i are the same.

5. Idempotent if $x_1 = x_2 = \dots = x_n = x$

$$m(\mathbf{x}, \mathbf{w}) = x.$$

6. Self duality properties of symmetric sums (or stability for the standard negation)

$$m(n(x), n(y)) = n(m(x, y))$$

where the negation is the strict negation introduced in bounded logic. This means that the reversal of the scale has no effects on the evaluations, and express self-duality. Silvert [31] introduced first this property and Dombi [7] studied whether the operator is associative.

7. Bisymmetry (also called mediality). If the weights are equal then

$$m(m(x_1, x_2), m(x_3, x_4)) = m(m(x_1, x_3), m(x_2, x_4)).$$

Bisymmetry expresses that the aggregation of all elements of any square matrix can be performed first on rows, then on columns or conversely.

$$\begin{aligned} m(m(x_{11}, \dots, x_{1n}), \dots, m(x_{n1}, \dots, x_{nn})) &= \\ &= m(m(x_{11}, \dots, x_{n1}), \dots, m(x_{1n}, \dots, x_{nn})) \\ x &= \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \end{aligned}$$

8. Autodistributivity

$$\begin{aligned} m(x_1, m(x_2, x_3)) &= m(m(x_1, x_2), m(x_1, x_3)) \\ m(m(x_1, x_2), x_3) &= m(m(x_1, x_3), m(x_2, x_3)) \end{aligned}$$

9. Decomposability. Let $a = m(x_{j+1}, x_{j+2}, \dots, x_k)$ then

$$m(x_1, x_2, \dots, x_n) = m(x_1, \dots, x_k, \underbrace{a, a, \dots, a}_{(k-j)}, x_{k+1}, \dots, x_n)$$

We can characterize the bounded mean with the following properties.

Theorem 3. *If m is continuous, strictly monotonous, idempotent, compatible, the bisymmetry holds, symmetric and the self duality is valid, then*

$$m(x, y) = f^{-1} \left(\frac{1}{2} (f(x) + f(y)) \right)$$

where $f : [0, 1] \rightarrow [0, 1]$ continuous, strictly monotonous function.

The proof is easy and based on Aczel's theorem [1] and from the Aczel theorem we get that if m is continuous, strictly monotonous and bisymmetric then

$$m(x_1, x_2, \dots, x_n) = f^{-1} \left(\sum_{i=1}^n w_i f(x_i) \right).$$

From the symmetry we get:

$$m(x, y) = f^{-1} \left(\frac{1}{2} (f(x) + f(y)) \right).$$

From the self duality using the substitution $y = n(x)$

$$n(m(x, n(x))) = m(x, n(x)).$$

This means that $m(x, n(x)) = \nu_*$

$$f^{-1} \left(\frac{1}{2} f(x) + \frac{1}{2} f(n(x)) \right) = \nu_*.$$

From this the negation:

$$n(x) = f^{-1} (2 \cdot f(\nu_*) - f(x)).$$

4.2 Preference Relation

Very often it is referred as valued preference modelling. In classical theory the preference is a binary relation. In the fuzzy theory preference is closely related to the implications. In the bounded system, we distinguish between logic (and, or, implication) and multicriteria decision tools (mean, preferences). The meaning of preference is

$$p(x, y) = \text{truth of } (x < y)$$

As we have seen at the generalized conjoint measurement p is a continuous function and strictly increasing in the second variable and decreasing in the first one.

In multicriteria decision-making the usual way is aggregating values or preferences building the weighted sum. In our bounded system the definition of the preference is based on this aggregation (i.e. or mean operator)

$$p(x, y) = m(n(x), y) = f^{-1} \left(\frac{1}{2}(f(y) - f(x)) + f(\nu_*) \right)$$

where ν_* is the neutral value of the negation.

So the preference and aggregation are not independent. $p(x, y)$ has several good properties.

1. The meaning of ν_* at the preference relation

$$\begin{aligned} p(x, y) > \nu_* &\iff x < y \\ p(x, x) &= \nu_* \\ p(x, y) < \nu_* &\iff x > y \end{aligned}$$

2. The following identities hold:

$$\begin{aligned} p(x, y) &= n(p(y, x)) \\ p(x, y) &= p(n(y), n(x)) \\ p(x, y) &= n(p(n(x), n(y))) \end{aligned}$$

3. The preference relation is exchangeable with the aggregation

$$\begin{aligned} p(m(x_1, w_1; \dots; x_n, w_n), m(y_1, w_1; \dots; y_n, w_n)) &\equiv \\ &\equiv m(p(x_1, y_1), w_1; \dots; p(x_n, y_n), w_n) \end{aligned}$$

The proof of property 1 and 2 is trivial. We will see the proof of property 3 later.

4. The preference relation is transitive in the following sense:

$$\text{If } p(x, y) > \nu_* \text{ and } p(y, z) > \nu_* \text{ then } p(x, z) > \nu_*.$$

Proof.

$$p(x, y) > \nu_* \iff \frac{1}{2}(f(y) - f(x)) > 0$$

$$p(y, z) > \nu_* \iff \frac{1}{2}(f(z) - f(y)) > 0$$

Adding the two equations we get

$$\frac{1}{2}(f(z) - f(x)) > 0$$

and from this

$$p(x, z) = f^{-1}\left(\frac{1}{2}(f(z) - f(x)) + f(\nu_*)\right) > f(\nu_*).$$

In the classical literature [19] $p(x, y)$ means that x is not worse than y , $p(x, y) = \text{truth}(x \geq y)$. From this $p(x, x) = 1$ and known $p(x, y)$ we can say nothing on $p(y, x)$, and this is the reason why transitivity is not an easy question. We know that this preference structure is primitive, but very useful. Introducing the modifications we can describe very sophisticated preferences and decisions.

4.3 Hedges (Unary Operators)

The so called hedges play an important role in fuzzy systems. Since Zadeh introduced this modifier, only few articles dealt with this concept, and no theoretical investigation can be found. The modifier “very” is $h(x) = x^2$ and “more or less” is $h(x) = x^{1/2}$. The concept is independent from other connectives. In a bounded system the hedges has the following form:

$$h^{(\lambda)}(x) = f^{-1}[\lambda f(x) + \alpha].$$

This unary operator involves the negation ($\alpha = 1, \lambda = -1$). Hedges can be built using conjunction and disjunction operators (see Fig. 1):

$$h_c^{(n)}(x) = c(x_1, \dots, x_n) = f^{-1}[nf(x) - (n - 1)],$$

$$h_d^{(n)}(x) = d(x_1, \dots, x_n) = f^{-1}[nf(x)].$$

The semantic meaning of the conjunctive hedge is “very” and of the disjunctive hedge is “more or less”. From the hedges one can build other hedges using the operations of composition and logical connectives. These play an important role:

$$h(x), \quad n(h(x)), \quad h(n(x)), \quad n(h(n(x))).$$

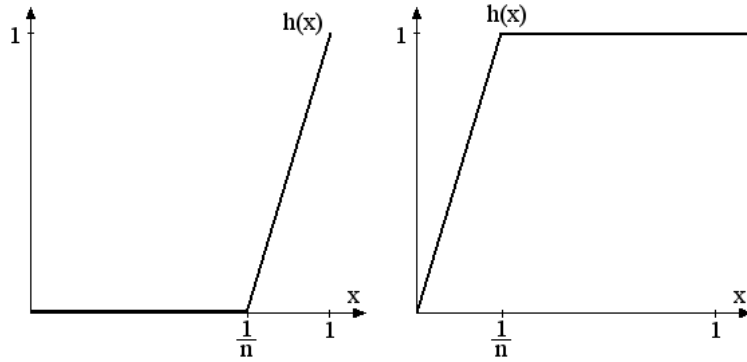


Fig. 1. (a) Conjunctive and (b) disjunctive hedge

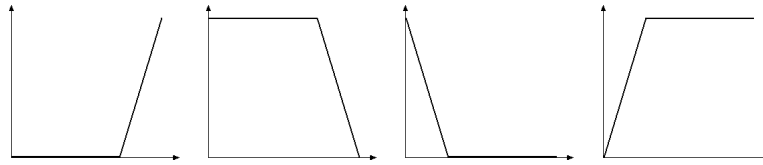


Fig. 2. (a) “good”, (b) “not good”, (c) “bad” and (d) “not bad” unary operators

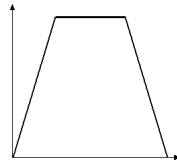


Fig. 3. “middle” unary operator

If $h(x)$ has the meaning of “very good” (preferable) then $n(h(x)) =$ “not(very good)”, $h(n(x)) =$ “bad” and $n(h(n(x))) =$ “not bad”, see Fig. 2.

Using the conjunction operator for “not good” and “not bad” we get the middle unary operator, see Fig. 3.

In a similar way the indifference relation can be constructed from the preference relation. It can be shown that only using the mean operator and the hedge functions we can get all different types of connectives. Let $h_c(x) = f^{-1}[2 \cdot f(x) - 1]$ then:

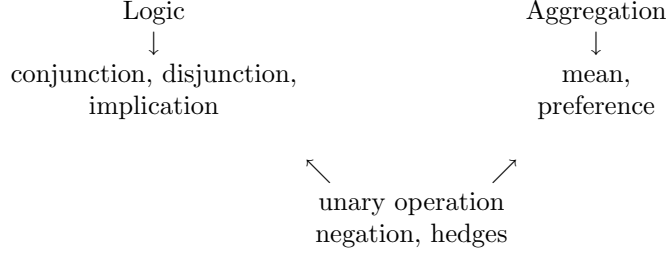
$$c(x, y) = h(m(x, y)).$$

We’ll also need a special unary operator:

$$h^{(\infty)}(x) = \begin{cases} 1 & \text{if } x > 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 0 & \text{if } x < 1/2 \end{cases}$$

4.4 General Overview of the Bounded System

It has two main points:



4.5 Special Bounded Systems

We distinguish the systems on the type of generator functions.

- If $f(x)$ is linear, then it is called Linear Bounded System (LBS)
- If $f(x)$ is polynomial, then it is called Polynomial Bounded System (PBS)
- If $f(x)$ is rational, then it is called Rational Bounded System (RBS)

Operators of the LBS system

$$\begin{aligned}
 c(x, y) &= [x + y - 1] & n(x) &= 1 - x \\
 d(x, y) &= [x + y] & \tau(x) &= [\lambda x + \alpha] \\
 i(x, y) &= [y - x + 1] \\
 m(x, y) &= \frac{1}{2}(x + y) \\
 p(x, y) &= \frac{1}{2}(y - x + 1)
 \end{aligned}$$

The generator function of PBS (Yager) is

$$f(x) = x^\gamma.$$

The generator function of RBS (Dombi) is

$$f(x) = \frac{1}{1 + k((1 - x)/x)^\gamma}.$$

5 Conjoint Measurement and Bounded System

In section 2 we dealt with conjoint measurement models. We can get the same models using preference relation and unary operators of the Bounded System. What we need is a threshold value. In Bounded System it is natural to choose it $1/2$.

Model (4) is equivalent with

$$p\left(\frac{1}{n}\sum_{i=1}^n u_i(x_i); \frac{1}{n}\sum_{i=1}^n u_i(y_i)\right) \geq \frac{1}{2}$$

Model (5) is equivalent with

$$\sum_{i=1}^n w_i \tau_i(p(x_i, y_i)) \geq \frac{1}{2}$$

It is an open question that the general models (3) or (6) can be built only by using elements of the Bounded System. On the other hand Bounded System could be more general, because non-monotonic operators could be used.

If interaction between the criteria appears, then the logic part of the Bounded System seems to be very useful. If decision rules are given, then also the logic part can be used.

6 Utility and Preference Based Decision Making

The most widespread methods used in multicriteria decision analysis are mainly of two types: the ones based on utility theory or on preference relations. Naturally, different methods provide different results even if the same information is used. The immediate question arises under which condition do the two approaches give the same result. More precisely, let a and b two actions (alternatives) and c_1, c_2, \dots, c_n different criteria, x_i and y_i are the utilities (evaluations) of a and b from the perspective of c_i , respectively.

Table 1.

| | c_1 | c_2 | \dots | c_n |
|-----|-------|-------|---------|-------|
| a | x_1 | x_2 | \dots | x_n |
| b | y_1 | y_2 | \dots | y_n |
| | w_1 | w_2 | \dots | w_n |

Let w_1, w_2, \dots, w_n denote the nonnegative weights of the criteria such that they are normalized, i.e.:

$$\sum_{i=1}^n w_i = 1$$

The aggregated utility of actions a and b are

$$A = U(a) = \sum_{i=1}^n w_i x_i \quad , \quad B = U(b) = \sum_{i=1}^n w_i y_i$$

The usual ordering of the numbers A and B induces an ordering of actions a and b (generally an ordering on the set of the actions).

On the other hand, in the preference-based methods first a and b are compared from the perspective of each c_i . This is usually done by calculating a preference index p_i

$$p_i = p(x_i, y_i)$$

with the help of some function p . The next step is the aggregation of the p_i -s:

$$P(a, b) = \sum_{i=1}^n w_i p(x_i, y_i)$$

To be able to compare the two approaches let us use the same function p and the aggregated utility values A and B to obtain another preference index $p(A, B)$ of actions a and b .

Theorem 4.

$$P(a, b) = p(A, B) \quad \text{iff} \quad p(x, y) = \frac{1}{2}(y - x + 1) \quad (7)$$

where $p(x, y)$ is the linear form of the universal preference function.

Proof.

$$\begin{aligned} p(A, B) &= \frac{1}{2}(B - A + 1) = \frac{1}{2} \left(\sum_{i=1}^n w_i y_i - \sum_{i=1}^n w_i x_i + \sum_{i=1}^n w_i \right) = \\ &= \sum_{i=1}^n w_i \frac{y_i - x_i + 1}{2} = \sum_{i=1}^n w_i p(x_i, y_i) = P(a, b). \quad \square \end{aligned}$$

Another way of comparing the two approaches is by calculating for each action the value

$$V(a) = \frac{1}{n} \sum_b P(b, a)$$

where n is the number of actions, and p is the preference function given in the above theorem.

Theorem 5.

$$V(a) > V(b) \quad \text{iff} \quad U(a) > U(b). \quad (8)$$

Proof. Let a_1, a_2, \dots, a_m alternatives, c_1, c_2, \dots, c_n criteria and e_{ki} are utilities of a_k from the perspective of c_i , respectively. For $V(a_k)$ we get:

$$\begin{aligned} V(a_k) &= \frac{1}{n} \sum_{x=1}^n p(a_k, a_x) = \frac{1}{n} \sum_{x=1}^n \sum_{i=1}^m w_i \frac{e_{ki} - e_{xi} + 1}{2} = \\ &= \frac{1}{2n} \sum_{x=1}^n \sum_{i=1}^m w_i (e_{ki} - e_{xi} + 1) \end{aligned} \quad (9)$$

and for

$$\begin{aligned} V(a_k) - V(a_l) &= \frac{1}{2n} \sum_{x=1}^n \left(\sum_{i=1}^m w_i ((e_{ki} - e_{xi} + 1) - (e_{li} - e_{xi} + 1)) \right) = \\ &= \frac{1}{2n} \sum_{x=1}^n \left(\sum_{i=1}^m w_i e_{ki} - \sum_{i=1}^m w_i e_{li} \right) = \frac{1}{2} (U(a_k) - U(a_l)). \square \end{aligned} \quad (10)$$

The function in (7) has some useful properties:

$$\begin{aligned} p(1, 0) &= 0, \quad p(0, 1) = 1, \quad p(x, x) = \frac{1}{2}, \\ p(x, y) + p(y, x) &= 1. \end{aligned}$$

Let the negation function $n(x) = 1 - x$. Then

$$p(x, y) + p(n(x), n(y)) = 1.$$

The next result gives the generalization of the theorem, using the aggregation operator

$$A = f^{-1} \left(\sum_{i=1}^n w_i f(x_i) \right)$$

axiomatized by Dyckhoff [11] and Fodor [18], where f is the generator function of the aggregation, $f : [0, 1] \rightarrow [0, 1]$ continuous and strictly monotone increasing. The generalized negation function is:

$$n(x) = f^{-1}(1 - f(x)) \quad (11)$$

Theorem 6.

$$\text{If } p^*(x, y) = f^{-1} \left(\frac{1}{2}(f(x) - f(y) + 1) \right) \text{ then } p^*(a, b) = p(A, B) \quad (12)$$

where $p^*(x, y)$ is the universal preference function.

In other words, the preference and the aggregation are interchangeable.

$$p(U(a), U(b)) = U(p(a, b)).$$

Proof.

$$\begin{aligned}
p^*(a, b) &= f^{-1} \left(\frac{1}{2} (f(b) - f(a) + 1) \right) = \\
&= f^{-1} \left(\frac{1}{2} \left(f \left(f^{-1} \left(\sum_{i=1}^m w_i f(y_i) \right) \right) - \right. \right. \\
&\quad \left. \left. - f \left(f^{-1} \left(\sum_{i=1}^m w_i f(x_i) \right) \right) + 1 \right) \right) = \\
&= f^{-1} \left(\frac{1}{2} \left(\sum_{i=1}^m w_i f(y_i) - \sum_{i=1}^m w_i f(x_i) + \sum_{i=1}^m w_i \right) \right) = \\
&= f^{-1} \left(\frac{1}{2} \sum_{i=1}^m w_i (f(y_i) - f(x_i) + 1) \right). \square
\end{aligned}$$

The properties (8), (9), (10) hold for p^* as well with the modification

$$p^*(x, x) = \nu,$$

where ν is the neutral value of $n(x)$, i.e. $n(\nu) = \nu$.

7 Outranking Approach Based on the Universal Preference Function

In “The Outranking Approach and the Foundation of ELECTRE Methods” [30], Bernard Roy presents the fundamental aspects of this approach: the basic principles in the construction of the outranking relation, the concepts of the concordance and discordance. In this paper we are concentrating only to the concordance relation. For validating a comprehensive outranking relation S , it is necessary to take into account the fact that the role which has to be devoted to each criterion in the aggregation procedure is not necessarily the same. In other words, we need to characterize what is usually referred to as “the greater or less importance” give to each criterion. In methods, the importance of the j -th criterion is taken into account. By definition, the concordance index $c(a, b)$ characterizes the strength of the positive arguments able to validate the assertion aSb . By definition in ELECTRE I method

$$c_1(a, b) = \sum_{j \in aSb} w_j$$

and in ELECTRE II method

$$c_2(a, b) = \sum_{j \in aSb} w_j \frac{p_j + q_j(a) - g_j(b)}{p_j - q_j}$$

where

$$\sum_{i=1}^n w_i = 1$$

The outranking method of the PROMETHEE method is quite similar to the concordance index in the ELECTRE method. In order to estimate the preferences the decision maker is offered a choice, for each criterion between six forms of curves presented in [5]. In function of the way his preference increase with difference $b-a$, the decision maker sets, for each criterion, the form of the function and associated parameter(s). The parameters to be estimated have simple interpretation since they are indifference and preference thresholds.

The outranking methods are not preference-based methods in the above sense (i.e. universal preference function), because in the ELECTRE methods [30]

$$p(x, x) = 1$$

and in the PROMETHEE methods [5]

$$p(x, x) = 0.$$

According to (11) and (12) PROMETHEE and ELECTRE seem to be dual approaches. Furthermore, $p(x, y)$ and $p(y, x)$ are not related. The main reason is that $p(x, y)$ differs from the form (7) and the function $p(x, y)$ varies from criterion to criterion, i.e.

$$p(a, b) = \sum_{i=1}^n w_i p_i(x_i y_i).$$

We define the general form of the outranking and the utility approach:

$$p(a, b) = \sum_{i=1}^n w_i \tau_i(p_i(x_i y_i)). \quad (13)$$

where $\tau_i : [0, 1] \rightarrow [0, 1]$ are unary functions.

The next result presents the two outranking methods in a unified way.

Theorem 7. *For the preference function p^{EL}, p^{PR} of the ELECTRE and PROMETHEE methods there exist unary functions τ_i^{EL} and τ_i^{PR} such that*

$$p^{EL}(a, b) = \sum_{i=1}^n w_i \tau_i^{EL}(p(x_i, y_i))$$

$$p^{PR}(a, b) = \sum_{i=1}^n w_i \tau_i^{PR}(p(x_i, y_i))$$

and moreover in the sense of theorem 5 we can get the utility approach from the general form (13).

Proof. Let $\tau^{EL}(x)$ and $\tau^{PR}(x)$ be the following functions:

$$\tau_i^{EL}(x) = \left[\frac{x - p_i}{q_i - p_i} \right]$$

for some $0 \leq p_i \leq q_i \leq \frac{1}{2}$, and

$$\tau_i^{PR}(x) = \left[\frac{x - p_i}{q_i - p_i} \right] = [\varphi(x)]$$

for some $\frac{1}{2} \leq p_i \leq q_i \leq 1$.

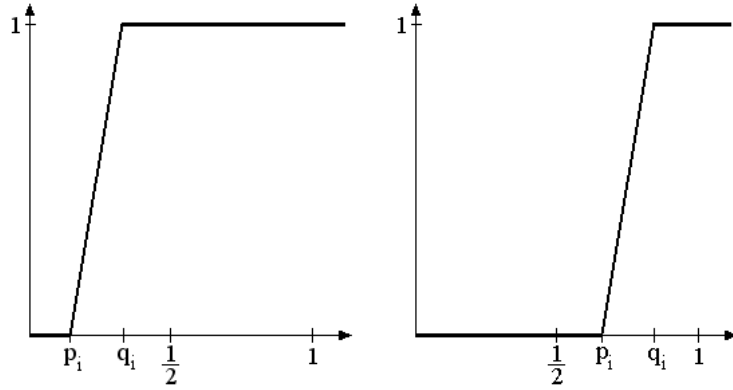


Fig. 4. τ^{EL} and τ^{PR} functions

The function $\varphi(x)$ might have different forms, see PROMETHEE [5].
The utility methods can be obtained by

$$\tau_i(x) = x.$$

We remark that if the τ function commute with the negation function i.e.

$$\tau_i(n(x)) = n(\tau(x))$$

then the properties (8), (9), (10) remain valid for p^* .

These constructions can be used to build a more general decision method choosing less restrictive functions for $\tau(x)$. It is incorporated into the Dec.Art method.

8 Lexicographic Decision Model

The lexicographic method is usually used if the values of the criteria are discrete, but it can be used also, when the values are numerical.

For the construction of the lexicographic method we need the preference relation, the $H^{(\infty)}(x)$ hedges and a special weighting system. Let

$$w_i = \frac{1}{2^i} + \frac{1}{n2^n}$$

where n is the number of the criteria. Then

$$\sum_{i=1}^n w_i = 1.$$

We construct the lexicographic order of all alternatives, aggregating all preference relations. Let a_1, \dots, a_n be the alternatives, c_1, \dots, c_n be the criteria and $x_{ij} \in [0, 1]$ be the corresponding value for a_i and c_j . Define

$$l_i = \frac{1}{m} \sum_{j=1}^m \left(1 - h^{(\infty)} \left(\sum_{k=1}^n w_k h^{(\infty)}(p(x_{ik}, x_{jk})) \right) \right)$$

where $l_i \in [0, 1]$ and gives back the lexicographic order of the (a_1, \dots, a_n) alternatives. The validity of the construction can be found in [10]. We mention only that

$$\sum_{k=1}^n w_k h^{(\infty)}(p(x_{ik}, x_{jk}))$$

is similar to binary number, which can be ordered lexicographically.

9 Assessment Methods

The most crucial question is how can we get the parameters of the decision procedures (i.e. weights, thresholds, function parameters etc.). In Bounded System we use only a few blocks and every block has semantical meaning. We can easily formulate our requirements. It could be a logic based filtering, aggregation of same criteria, even special lexicographic decision making as a subset of criteria. This means, that heterogenous decision model can be handled. The real world problems have this nature.

The assessment is possible by learning the parameters. Getting information on historical data or from experts, showing them a virtual situation, the model can be modified and with changing the parameters the identification process can be carried out.

It is worth to be mentioned, that the model can be translated into a neural network, and the well known neural learning algorithms are applicable.

The so called decision making by using rules method also belongs to the inherent concept of Bounded System. Greco, Matarazzo and Slowinski [23] investigate preference relation to the conjoint measurement and shows the equivalences of the two method.

10 Conclusions

We presented Bounded System, which is based on the nilpotent operator and is extended by mean, preference operator and hedges, consistent with the logic part. We showed that various decision models can be described with the bounded system as MAUT, PROMETHEE, ELECTRE and the lexicographic decision rule. We specified the general function of the conjoint measurement without the loss of generality. The two worlds, logic and aggregation were constructed in a consistent way. With Bounded System a heterogeneous decision system can be built up i.e. on some criteria we can make a lexicographic decision, on some others we can use an outranking approach with veto threshold and maybe on the remaining criteria the utility approach.

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