

Inference Models

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In this paper a consistent model of the generalized modus ponens is discussed in the approximate reasoning. Inputs are implication rules (the so-called: IF-THEN structures) and statements. Output is a statement (conclusion). Approximate reasoning allows fuzzy inputs, fuzzy antecedents, fuzzy consequents, or combinations of these. In general in the approximate reasoning the continuous logic is used instead of classical logic.

One of the most successful area of fuzzy theory is the fuzzy control based on inference, although from logical point of view only few theoretical papers

deal with it. The problem is, that there are several operator systems in fuzzy logic (conjunction, disjunction, negation) and there are a lot of implication. The so-called modifiers also play an important role and they still no have any axiomatic bases. The inference is an algorithm to calculate the unknown values. But what kind of inference is good?

Fuzzy theory deals with uncertain information, but in control the input and the result are not uncertain. (So the membership function should have a different meaning, etc.)

As in control area the only strict monotonously increasing operator is useful. We are concentrating on only those operators that come from this class. The theoretical basis of the modifiers is also given. The crucial point is choosing the implication, because the residual implication is non-continuous.

Our solution is using the classical extension with approximate properties. We use different models for the inference: solving equations, optimization and surface approximation. We show that all of them is consistent with the classical inference.

Inference:

Modus Ponens (MP)

$$\frac{x \quad x \longrightarrow y}{x}$$

$$\frac{x \quad x \wedge y, \quad \text{etc.}}{y}$$

Modus Tollens (MT)

$$\frac{\bar{y} \quad x \longrightarrow y}{\bar{x}}$$

implies



$$\begin{aligned}(x \wedge (x \longrightarrow y)) &\longrightarrow y = 1 \\ (\bar{y} \wedge (x \longrightarrow y)) &\longrightarrow \bar{x} = 1 \\ (x \wedge (x \wedge y)) &\longrightarrow y = 1\end{aligned}$$

Two levels



and

Approximate reasoning

Generalized modus ponens (GMP)

$$\frac{\text{If } x \text{ is } A \text{ THEN } y \text{ is } B}{x \text{ is } A'} \quad y \text{ is } B'$$

Generalized modus tollens (GMT)

$$\frac{\text{If } x \text{ is } A \text{ THEN } y \text{ is } B}{y \text{ is } B'} \quad x \text{ is } A'$$

Implication

Modification

Implication $x \longrightarrow y$

a) identity principle

$$x \longrightarrow y = 1 \iff x \leq y$$

b) exchange principle

$$x \longrightarrow (y \longrightarrow z) = y \longrightarrow (x \longrightarrow z)$$

c) contrapositive symmetry

$$x \longrightarrow \bar{y} = y \longrightarrow \bar{x}$$

Residual implication and Modus ponens:

MP:

$$(x \wedge (x \longrightarrow y)) \longrightarrow y = 1 \iff x \wedge (x \longrightarrow y) \leq y$$

$$x \wedge \underset{x \longrightarrow y}{z} \leq y$$

$$\sup_z (x \wedge z \leq y) = x \longrightarrow y$$

Associativity from exchange principle and contrapositive symmetry:

$$x \circ y := \bar{x} \longrightarrow y$$

$$\begin{aligned}(x \circ y) \circ z &= (\overline{\bar{x} \longrightarrow y}) \longrightarrow z = \bar{z} \longrightarrow (\bar{x} \longrightarrow y) = \\ &= \bar{x} \longrightarrow (\bar{z} \longrightarrow y) = \bar{x} \longrightarrow (\overline{\bar{y} \longrightarrow z}) = x \circ (y \circ z)\end{aligned}$$

Corollary 1

$$x \longrightarrow y = f_d^{-1}(f_d(n(x)) + f_d(y))$$

Implication Nilpotent class
Lukasiewicz logic

$$x \longrightarrow y = [y - x + 1]$$

where

$$[x] \begin{cases} 1 & \text{if } x > 1 \\ x & \text{otherwise} \\ 0 & \text{if } x < 0 \end{cases}$$

Min, max operator class

Strict monotonously increasing class

Inequalities and logical expressions:

Disjunction

$$\overline{P_1} \vee P_2 \vee \overline{P_3}$$
$$\Downarrow$$

$$1 - x_1 + x_2 + 1 - x_3 \geq 1 \quad x_1, x_2, x_3 \in \{0, 1\}$$

$$-x_1 + x_2 - x_3 \geq 1 - 2$$

General

$$c_1x_1 + c_2x_2 + \dots + c_nx_n \geq 1 - n(c)$$

$n(c)$ is the number of negation $c_i \in \{0, 1\}$

Conjunction

$$P_1 \wedge \overline{P_2}$$

$$\left. \begin{array}{l} P_1 \geq 1 \\ 1 - P_2 \geq 1 \end{array} \right\} \text{System of inequalities}$$

Conjunctive normal form:

$$(x_1 \vee x_3) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$$

Representations:

$$\begin{array}{rcl} x_1 + x_3 & \geq & 1 \\ -x_2 + x_3 & \geq & 0 \\ -x_1 + x_2 - x_3 & \geq & -1 \end{array} \quad x_i \in \{0, 1\}$$

Feasibility

Inference:

$$\begin{aligned}\bar{x}_1 \vee \bar{x}_3 &= 1 \\ x_1 \vee x_2 \vee \bar{x}_3 &= 1 \\ x_1 \vee \bar{x}_2 \vee \bar{x}_3 &= 1\end{aligned}$$

$$\begin{array}{c} \text{-----} \\ \bar{x}_3 \end{array}$$

Representation:

$$\begin{aligned}-x_1 - x_3 &\geq 1 - 2 \\ x_1 + x_2 - x_3 &\geq 1 - 1 \\ x_1 - x_2 - x_3 &\geq 1 - 2\end{aligned}$$

$$\begin{array}{c} \text{-----} \\ -\bar{x}_3 = Z \longrightarrow \min \end{array}$$

Inference general

$$\begin{array}{rcl}
 \bigvee_{j \in K_1} \pm P_{1,j} & c^{(1)}(x) \geq & 1 - n(c^{(1)}) \\
 \bigvee_{j \in K_2} \pm P_{2,j} & c^{(2)}(x) \geq & 1 - n(c^{(2)}) \\
 \bigvee_{j \in K_n} \pm P_{n,j} & c^{(n)}(x) \geq & 1 - n(c^{(n)}) \\
 & & \text{-----} \quad \text{-----} \\
 \bigvee_{j \in K_{n+1}} \pm P_{n+1,j} & c^{(n+1)}(x) = z \longrightarrow & \min
 \end{array}$$

Logically implied (Inference)

LP optimum: mindestens $1 - n(c^{(n+1)})$
 (Or the optimum greater than $-n(c)$!)
 \Downarrow
 inference is valid

Other model

a)

$$\begin{aligned}x_0 + x_1 + x_3 &\geq 1 \\x_0 - x_2 + x_3 &\geq 0 \\x_0 - x_1 + x_2 - x_3 &\geq -1\end{aligned}$$

$$\min x_0$$

$$\text{-----}$$
$$x_0 = 0 \text{ then feasible}$$

b)

$$\begin{aligned}S_1 \text{ CNF } Ax &\geq b \text{ and } (S_1 \text{ is consistent}) \\S_2 \text{ CNF } Bz &\geq d\end{aligned}$$

S_2 is logically implied by S_1

$$\begin{aligned}x_0 e + Bx &\geq d \quad x \in \{0, 1\} \\Ax &\geq b\end{aligned}$$

$$\min x_0$$

S_1 implied $S_2 \iff x_0 = 1$

The modus ponens case

$$\frac{x}{\frac{x \longrightarrow y}{y}}$$

$$x \longrightarrow y = \bar{x} \vee y \implies \underbrace{1 - x + y}_{\geq 1}$$

Nilpotent

↓

$$\text{Identity principle} \quad \longleftarrow \quad x \leq y$$

$$\frac{x \geq 1}{\frac{x \leq y}{y \implies \min} \quad y = 1}$$

Graphical representation

Modus ponens on an interval

$$x \in A$$

$$X \in A \text{ then } y \in B$$

$$X_A(x) = 1$$

$$X_A(x) \longrightarrow X_B(x)$$

$$X_A(x) > y$$

$$X_A(x) \leq X_B(x)$$

Graphical representation:

Implication \implies Intersection

Fuzzy case: α - cut

$$(\mu_A(x) > \delta \wedge \mu_B(x) > \delta) = 1$$

x an object A and B properties

Inference!

Changing the restriction:

Rectangular case:

Fuzzy case: Characterization with the highest value

$$F(x, y) = (\mu_A \cap \mu_B \cap \mu_C)$$

$$F(x_a, y_a) = A$$

$$F(x_b, y_b) = B$$

$$A \gg B$$

Rule system:

- R_1 : if $x_1 \in A_{11}$ and $x_2 \in A_{12}$ and ... and $x_n \in A_{1n}$ then $y \in B_1$
 R_2 : if $x_1 \in A_{21}$ and $x_2 \in A_{22}$ and ... and $x_n \in A_{2n}$ then $y \in B_2$
:
 R_m : if $x_1 \in A_{m1}$ and $x_2 \in A_{m2}$ and ... and $x_n \in A_{mn}$ then $y \in B_m$

$$x_1 \in A_1^*, x_2 \in A_2^*, \dots, x_n \in A_n^* \text{ then } y \in B^*$$

Modeling with $\sigma(x)$ function:

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}}$$

$$\sigma_a^{(\lambda)}(x) = \text{truth}(a \leq x)$$

$$x \in (a, b) \iff (\text{truth}(a \leq x)) \text{ and } \text{truth}(x \leq b)$$

Dombi operator

$$o_\alpha(x, y) = \frac{1}{1 + \left(\left(\frac{1-x}{x} \right)^\alpha + \left(\frac{1-y}{y} \right)^\alpha \right)^{\frac{1}{\alpha}}}$$

Match of the the Rule (i)

$$(x_k \in A_{ik}) \text{ and } (x_k \in A_k^*)$$

$$\begin{aligned} a_{ik} &\leq x_k \leq b_{ik} \\ a_k^* &\leq x_k \leq b_k^* \end{aligned}$$

$$a \leq b \leq c \leq d$$

$$a, b, c, d \in \{a_{ik}, b_{ik}, a_k^*, b_k^*\}$$

$\lambda_a, \lambda_b, \lambda_c, \lambda_d$ the sharpness (certainty factors)

Highest value:

$$x_{k \max}^* = \frac{b\lambda_b + c\lambda_c + (\ln \lambda_b - \ln \lambda_c)}{\lambda_a + \lambda_b}$$

$$r_k^* = c(x_{1, \max}^*, x_{2, \max}^*, \dots, x_{n, \max}^*)$$

$$r_k = \frac{r_k^*}{\sum_{k=1}^m r_k^*}$$

Consequence:

$$\sigma_{a^*, b^*}^{\lambda_a^*, \lambda_b^*}(x) = ?$$

$$a^* = \sum r_i a'_i \quad b^* = \sum r_i b'_i$$

where

$$B_i = [a_i b_i]$$

$$\lambda_{a^*} = ? \quad \lambda_{b^*} = ?$$

$$\sigma_a^{\lambda_a}(x), \sigma_b^{\lambda_b}(x) \implies \sigma_{a,b}^{(\lambda)}(x)$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_a} + \frac{1}{\lambda_b}$$

$$\frac{1}{\lambda_a^*} = \sum_{i=1}^m \frac{r_k}{\lambda'_{ai}}$$

$$\frac{1}{\lambda_b^*} = \sum_{i=1}^m \frac{r_k}{\lambda'_{bi}}$$

Defuzzification

$$x_m$$