DeMorgan Class and Negation

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Abstract— **The study in this paper is a theoretical inquiry into new representation of negation operators.**

New fuzzy negation operator family is proposed, which is based on theoretical investigation. Using new negation new t-norm and t-conorm can be formed too.

Our starting point is the necessity and sufficient condition of the DeMorgan law. On the basis of this result we establish a new condition of negation. We show that the well-known negations can be derived by using the representation theorem.

The famous result of Trillas on negation gives a representation theory too. The generator function of the negation corresponds the nilpotent class of t-norms and t-conorms. Our result is the representation theorem of negation in case of strict t-norm and t-conorm. From this investigation we get the theoretical background of Roychowdhury's result and it is generalized too.

Keywords— **Fuzzy sets, complement, t-norm, t-conorm**

I. Introduction

Zadeh in his seminal paper [16] on fuzzy sets defined an unary operator called the complement of fuzzy sets. In fuzzy set theory this operator plays such important role as the complement operator in the classical set theory. Without this complement operator we cannot describe any closed form of algebraic or logic system [10]

In a classical set the complement generators are categories of white and black (true, false). When these binary truth values are extended to multiple truth values, then the straight hard-line separation becomes fuzzy. The interpretation of complement functions focuses not only on the elements themselves, but on the membership (belongness) of the elements too. See Lukasiewicz [9] three-valued sets or other works [10][11]

This interpretation is definitely in contrast with the classical interpretation The fuzzy complement operator on fuzzy set builds another fuzzy set, which has different membership function. By changing the complement, the negated membership function also changes. In fuzzy logic we use the word "negation" instead of "complement".

II. Preliminaries

The fuzzy negation function has been studied by many researchers [8] [11][13][14][15] in 1980's.

The fuzzy complement operator has been associated with the lattice theory [2][7] and therefore the idea of order-reversal has naturally became an alterogative viewpoint of the definition.

However, up to now most of the existing practical fuzzy systems use Zadeh's standard fuzzy negation [16].

$$n(x) = 1 - x \tag{1}$$

This is the simplest negation function. Hamacher has shown, that it is the only form of negation which has polynomial form.

Yager [15] proposed the intuitinistic fuzzy negation function

$$n(x) = (1 - x^{\alpha})^{\frac{1}{\alpha}} \tag{2}$$

where $\alpha \in (0, \infty)$

Ovchinnikov [11] defined a dual of the intuitionistic complement.

Another interesting fuzzy negation function was proposed by Sugeno [13] in context of fuzzy measures and fuzzy integrals.

$$n(x) = \frac{1 - x}{1 + x} \tag{3}$$

Hamacher generalized Sugeno's result and got

$$n(x) = \frac{1 - x}{1 + \lambda x} \tag{4}$$

where $\lambda \in (1, -\infty)$. Hamacher has shown, that (4) is the general form of the negation belonging to the class of rational polynomials.

Both of the above mentioned classes (2) and (3) reduce to (1), when $\alpha = 1$, and $\lambda = 0$ respectively.

Dombi [4] gives the general form of the negation by studying the aggregation operator

$$n(x) = f^{-1}(2f(\nu_*) - f(x)) \tag{5}$$

where $\nu_* \in (0, 1)$ and $n(\nu_*) = \nu_*$, and f is the generator function of the aggregation operator. The rational form of this negation is

$$n(x) = \frac{\nu_*^2 (1 - x)}{\nu_*^2 (1 - x) + (1 - \nu_*)^2 x} \tag{6}$$

If $\nu_* = \frac{1}{2}$, then (5) reduce to (1), and if

$$\lambda = \left(\frac{1 - \nu_*}{\nu_*}\right) - 1$$

then (5) is equivalent to (3).

The negation of a fuzzy set is given by the following definition:

Definition 1: (Negation)
$n : [0, 1] \longrightarrow [0, 1]$ satisfying the following conditions:

C1: $n(0) = 1, n(1) = 0$ (Boundary conditions)
C2: $n(x) < n(y)$ for $x > y$ (Monotonicity)
C3: $n(x)$ is continuous (Continuity)
C4: $n(n(x)) = x$ (Involutivness)

From Trillas [14] result we know that negation can be established by generator function. It is the general representation theorem of fuzzy negation.

$$n(x) = f^{-1}(1 - f(x)) \tag{7}$$

when $f(x)$ is the generator function of the negation, $f : [0,1] \longrightarrow [0,1]$ continuous, strictly increasing and $f(0) = 0, \qquad f(1) = 1.$

This generator also generates the classes of Zadeh, Yager, Sugeno and Dombi.

Yager:

$$f(x) = x^{\alpha}, \tag{8}$$

Sugeno:

$$f(x) = \frac{1}{\lambda}\log(1 + \lambda x), \tag{9}$$

Dombi:

$$f(x) = \frac{(1-\nu)x}{(1-\nu)x + x(1-\nu)}. \tag{10}$$

Klir and Yuan [6] have also shown that involutive fuzzy negation functions can be generated by the dual of (6)

$$n(x) = f\left(1 - f^{-1}(x)\right) \tag{11}$$

where f is the same function, which was defined in (7).

III. Examination of the generalized DeMorgan identity

The condition of working with consistent fuzzy operators is the validity of some Bool identity. The most important one is DeMorgan law. Esteva and Dombi were the first two researchers, who carefully studied the DeMorgan identity. [5][4] It corresponds the conjunction, disjunction and negation. It is well-known, that conjunctive and disjunctive operators, which are strict monotonously increasing, associative, Archimedian, and fulfilling the boundary conditions have the following form:

$$c(x,y) = f_c^{-1}(f_c(x) + f_c(y)) \tag{12}$$

$$d(x,y) = f_d^{-1}(f_d(x) + f_d(y)) \tag{13}$$

where $f_c(x), f_d(x)$ the generator function of the conjunctive is respectively disjunctive operator. The shape of the function can be found in Dombi's paper [3], and the result is based on Aczél's theorem [1].

In the following, we examine the relation of the $f_c(x), f_d(x)$ and $n(x)$, if $c(x,y)$,$d(x,y)$ and $n(x)$ fulfills the DeMorgan law. First let's generalize the conjunction and disjunction operators.

Let

$$c(x_1, K_1; x_2, K_2; ...x_n, K_n) = f_c^{-1}\left(\sum_{i=1}^{n} K_i f_c(x_i)\right) \tag{14}$$

$$d(x_1, K_1; x_2, K_2; ...x_n, K_n) = f_d^{-1}\left(\sum_{i=1}^{n} K_i f_d(x_i)\right) \tag{15}$$

We call the (13) and (14) generalized operators $K_i \in \mathbb{R}$. The DeMorgan law for (13) and (14) is

$$c(n(x_1), K_1; n(x_2), K_2; ...n(x_n), K_n) =$$

$$= n(d(x_1, K_1; x_2, K_2; ...x_n, K_n)) \tag{16}$$

Theorem 2: (DeMorgan law)

Let c and d have the form (13),(14). The generalized DeMorgan is valid iff

$$f_c^{-1}(x) = n\left(f_d^{-1}(ax)\right) \tag{17}$$

where $a \neq 0$.

Proof: In the sequel of the proof we shall use the solution of the Cauchy functional equality, i.e

$$h(x+y) = h(x) + h(y) \tag{18}$$

h is continuous; The solution of the (17) is

$$h(x) = ax \qquad a \neq 0 \tag{19}$$

Description of DeMorgan law by using the generator function of conjunction and disjunction operators

$$f_c^{-1}\left(\sum_{i=1}^{n} K_i f_c(n(x_i))\right) =$$

$$= n\left(f_d^{-1}\left(\sum_{i=1}^{n} K_i(f_d(x_i))\right)\right) \tag{20}$$

Let's substitute $x_i^* = f_d(x_i)$ and $x_i = f_d^{-1}(x_i^*)$, so (18) can be formed:

$$f_c^{-1}\left(\sum_{i=1}^{n} K_i f_c\left(n\left(f_d^{-1}(x_i^*)\right)\right)\right) =$$

$$= n\left(f_d^{-1}\left(\sum_{i=1}^{n} K_i x_i^*\right)\right) \tag{21}$$

Let's use $f_c(x)$ for both side of equation (19) introducing the following notation:

$$h(x_i^*) = f_c\left(n\left(f_d^{-1}(x_i^*)\right)\right) \tag{22}$$

we get

$$\sum_{i=1}^{n} K_i h(x_i) = h\left(\sum_{i=1}^{n} K_i x_i\right) \tag{23}$$

This is the Cauchy equation. By using the result of it we get from (21)

$$ax^* = f_c\left(n\left(f_d^{-1}(x^*)\right)\right) \qquad (24)$$

Substituting $x = ax^*$ and applying f_c^{-1} for both side, we get the desired result (15). ■

Remark 3: On the basis of theorem 2, by the given $f_c(x)$ and $n(x)$, $f_d(x)$ can be determined, so that $f_c(x), f_d(x)$ and $n(x)$ is a DeMorgan triple. Similar to the above mentioned, by the given $f_d(x)$ and $n(x)$, $f_c(x)$ can be determined, so that $f_c(x), f_d(x)$ and $n(x)$ is a DeMorgan triple.

Remark 4: The proof of the theorem differs from [3], given by Dombi, but the result is the same.

IV. Negation and DeMorgan law

Naturally arises the question, if $f_c(x)$ és $f_d(x)$ are given, then what kind of condition ensure that $n(x)$ is negation (i.e. fullfils C1-C4). From theorem 2 we know that the necessity and sufficient condition of DeMorgan law is (17). Using the $x = f_c^{-1}(x)$ substitution from this, we get

$$x = n\left(f_d^{-1}(af_c(x))\right) \quad a \neq 0$$

Because $n(x)$ is involutive, we get

$$n(x) = f_d^{-1}(af_c(x)) \quad a \neq 0 \qquad (25)$$

This negation fulfills (C1-C3). The most important question is C4, the involutivness of (25): $n(x) = n^{-1}(x)$.

Theorem 5: (Involutive negation)
$n(x)$ given by (25) is involutive iff

$$f_c(x) = \frac{1}{a}k(f_d(x)) \quad a \neq 0 \qquad (26)$$

where $k : (0, \infty) \longrightarrow (\infty, 0)$ strictly decreasing function with the property

$$k^{-1}(x) = k(x) \qquad (27)$$

Proof: Involutivness means: $n^{-1}(x) = n(x)$, so from (25) we get

$$f_c^{-1}\left(\frac{1}{a}f_d(x)\right) = f_d^{-1}(af_c(x)) \qquad (28)$$

Let's find $f_c(x)$ in the following form

$$f_c(x) = h(f_d(x)) \qquad (29)$$

From this

$$f_c^{-1} = f_d^{-1}\left(h^{-1}(x)\right) \qquad (30)$$

Substituting (29) and (30) into (28) we get

$$f_d^{-1}\left(h^{-1}\left(\frac{1}{a}f_d(x)\right)\right) = f_d^{-1}(ah(f_d(x))) \qquad (31)$$

Let's use $f_d(x)$ for both side and using the notation $x = f_d(x)$, then

$$h^{-1}\left(\frac{1}{a}x\right) = ah(x)$$

Let $k(x) = ah(x)$, then from (29) we get the statement of the theorem (26) and it is also valid

$$k^{-1}(x) = k(x) \qquad (32)$$

■

From theorem 5 we can get a new representation theorem for the negation.

Theorem 6: (General form of negation.)
$f_c(x)$, $f_d(x)$, $n(x)$ is a DeMorgan triple iff

$$f_c(x) = \frac{1}{a}k(f_d(x)) \qquad (33)$$

and

$$n(x) = f^{-1}(k(f(x))) \qquad (34)$$

where $f(x) = f_c(x)$ or $f(x) = f_d(x)$ and $k(x)$ is a strictly decreasing function with property

$$k(x) = k^{-1}(x)$$

Proof: Because $f_c(x) = h(f_d(x)) = \frac{1}{a}k(f_d(x))$ and $n(x) = f_d^{-1}(af_c(x))$, so

$$n(x) = f_d^{-1}(k(f_d(x))) \qquad (35)$$

$n(x)$ can be expressed by $f_c(x)$. Using the fact, that $k(f_d(x)) = af_c(x)$ and $f_d^{-1}(x) = f_c^{-1}\left(\frac{1}{a}k^{-1}(x)\right) = f_c^{-1}\left(\frac{1}{a}k(x)\right)$, we get from (34)

$$n(x) = f_c^{-1}(k(f_c(x))) \qquad (36)$$

(33) is proved by (34) and (35). (32) is proved by theorem 5. It is easy to check, that C1-C4 are valid. ■

From (34) easy to get

$$k(x) = f\left(n\left(f^{-1}(x)\right)\right) \qquad (37)$$

i.e. if $f(x)$ and $n(x)$ is given, then $k(x)$ is determined by (37).

Another interesting question: Is (34) a general representation from the negation? The following theorem ensures, that all negation has (34) form.

While Trillas's theorem is negation for the nilpotent class of t-norm and t-conorm, our next result gives a representation theorem of the strict t-norm and t-conorm.

Theorem 7: (Representation theorem of negation)
For all given $n(x)$ and $k(x)$, there exist an $f(x)$

$$n(x) = f^{-1}(k(f(x)))$$

where $k(x)$ is a strictly decreasing function with property. $k(x) = k^{-1}(x)$ and $f(x)$ the generator function of conjunctive, or disjunctive operator.

Proof: We shall use the generalization of Trillas's result (see appendix) i.e. for every $n(x)$ there exist $f(x)$.,so

$$n(x) = f_*^{-1}(n_0(f_*(x))) \tag{38}$$

where $f_* : [0,1] \longrightarrow [0,1]$ continuously strictly increasing function and $n_0(x)$ is a negation. (See appendix)
Let

$$n_0(x) = f^{-1}(k(f(x))) \tag{39}$$

Substitute (38) into (37), then

$$n(x) = g^{-1}(k(g(x)))$$

where

$$g(x) = f(f_*(x))$$

■

Lemma 8: Zadeh's negation has an important role. If

$$k(x) = f\left(1 - f^{-1}(x)\right) \tag{40}$$

then

$$n(x) = 1 - x$$

Proof: From (33), we get (40). ■

Lemma 9: DeMorgan triple can be built using only one operator's generator function and choosing a $k(x)$: i.e. it is valid:

$$n(x) = f_c^{-1}(k(f_c(x))) \tag{41}$$

$$c(x,y) = f_c^{-1}(f_c(x) + f_c(y)) \tag{42}$$

$$d(x,y) = f_c^{-1}(k(k(f_c(x)) + k(f_c(y)))) \tag{43}$$

Proof: From DeMorgan law

$$\begin{aligned} d(x,y) &= n(c(n(x), n(y))) = \\ &= n\left(f_c^{-1}(f_c(n(x)) + f_c(n(y)))\right) \end{aligned}$$

Applying (33), we get (40). ■
Remark 10: Let's use for $k(x) = \frac{1}{x}$, then we get a special case of the result of Roychowdhury [12] and our theorem ensures that the system is DeMorgan.

V. Examples for DeMorgan systems

Using lemma 8, and lemma 9, we can get the classical and new operator systems.
1. If

$$\begin{aligned} f_c(x) &= -\ln(x) \\ n(x) &= 1 - x \end{aligned}$$

then

$$\begin{aligned} c(x,y) &= xy \\ d(x,y) &= x + y - xy \\ n(x) &= 1 - x \end{aligned}$$

2. On the other hand we can build new operators:
If

$$f_c(x) = -\ln(x)$$

$$k(x) = \frac{1}{x}$$

then

$$c(x,y) = xy$$

$$n(x) = e^{\frac{1}{\ln x}}$$

$$d(x,y) = e^{\left(\frac{1}{\ln\left(e^{\left(\frac{1}{\ln x} + \frac{1}{\ln y}\right)}\right)}\right)}$$

VI. Parameters of the negation

From [C1-C4] follows, that there exist ν_* fix point of negation

$$n(\nu_*) = \nu_* \tag{44}$$

It is another possible characterization of negation, if we give ν decision value for a given ν_0 (usually $\nu_0 = \frac{1}{2}$).
If x value is less than the decision value, the negated value is larger than the threshold and vice versa

$$\begin{aligned} x < \nu \quad &\textit{then} \quad n(x) > \nu_0 \\ x > \nu \quad &\textit{then} \quad n(x) < \nu_0 \end{aligned}$$

If $x = \nu$, then

$$n(\nu) = \nu_0 \tag{45}$$

If $n(x)$ has ν_* fix point, we use the notation $n_{\nu_*}(x)$ and if decision value is ν, then $n_\nu(x)$. Let's characterize the negation with ν_*, ν_0, ν parameters using that in (25) the negation has free parameter "a".

Lemma 11: The parametrial form of the negation is

$$n(x) = f^{-1}\left(f(\nu_*)\frac{k(f(x))}{k(f(\nu_*))}\right) \tag{46}$$

$$n(x) = f^{-1}\left(f(\nu_0)\frac{k(f(x))}{k(f(\nu))}\right) \tag{47}$$

Proof: Because from (25)

$$\nu_* = n(\nu_*) = f_d^{-1}(af_c(\nu))$$

expressing ν_* we get

$$a = \frac{f_d(\nu_*)}{f_c(\nu_*)}$$

in

$$n(x) = f_d^{-1}\left(f_d(\nu_*)\frac{f_c(x)}{f_c(\nu_*)}\right)$$

■

Using (33), we get (46). Using (28) similar result can be obtained with $f_c(x)$ generator function. So in (46) we can leave the index of the generator function. (47) can be proved in similar way.

VII. Reciprocal DeMorgan systems

From Dombi's result [3] we know, that if $f(x)$ is a generator function, then $f^{\gamma}(x)$ is a generator function too. As we see $k(x)$ plays an important role in the DeMorgan systems. Define the reciprocal system with the simplest one.

Definition 12: (Reciprocal system)

If

$$k(x) = \frac{1}{x} \quad \text{and} \tag{48}$$

$$f_{\gamma}(x) = f^{\,\gamma}(x) \tag{49}$$

we call it reciprocal system.

Remark 13: The first reciprocal system was introduced by Roychowdhury [12], who defines it only with (48).

Theorem 14: The general form of reciprocal DeMorgan system is

$$o_{\gamma}(x,y) = f^{-1}\left(f^{\,\gamma}(x) + f^{\,\gamma}(y)\right)^{\frac{1}{\gamma}} \tag{50}$$

$$n(x) = f^{-1}\left(f(\nu_0)\frac{f(\nu)}{f(x)}\right) \quad \text{or} \tag{51}$$

$$n(x) = f^{-1}\left(f(\nu_*)\frac{f^{\,2}(\nu_*)}{f(x)}\right) \tag{52}$$

where $f(x)$ is the generator function of either the conjunctive or the disjunctive operator. If $f(x) = f_c(x)$, then

$$\gamma = 1 \quad o_1(x,y) \quad = c(x,y) \tag{53}$$

$$\gamma = -1 \quad o_{-1}(x,y) \quad = d(x,y) \tag{54}$$

$$\gamma = \infty \quad o_{\infty}(x,y) \quad = \min(x,y) \tag{55}$$

$$\gamma = -\infty \quad o_{-\infty}(x,y) \quad = \max(x,y) \tag{56}$$

Proof: Using the $f_{\gamma}^{-1}(x) = f\left(x^{\frac{1}{\gamma}}\right)$ (53) is valid. From (26)

$$f_c(x) = \frac{1}{a}\,\frac{1}{f(x)}, \qquad f_c^{-1}(x) = f_d^{-1}\left(\frac{a}{x}\right)$$

so

$$\begin{aligned} d(x,y) &= f_d^{-1}(f_d(x) + f_d(y)) = \\ &= f_c^{-1}\left(\frac{1}{f_c(x)} + \frac{1}{f_c(y)}\right)^{-1} = \\ &= f^{-1}\left(f^{\,-\gamma}(x) + f^{\,-\gamma}(y)\right)^{-\frac{1}{\gamma}} \end{aligned}$$

(54) is valid. Using (49) and (50) and the reciprocal properties we get (51), (52).

$\gamma = -\infty$, and $\gamma = \infty$ case can be found in Dombi's article[3] ■

Remark 15: It is important, that in reciprocal system the negation is independent of the γ parameter i.e. independent of the value and the sign of γ. (In other words it is independent of the generator function, i.e. it belongs to conjunctive or disjunctive operator.)

Remark 16: The negation is independent of γ i.e. .the negation is the same for all γ

We can get the generalized Dombi operator system [4]

$$f(x) = \frac{1-x}{x} \tag{57}$$

then

$$o_{\gamma}(x_1, x_2, ..., x_n) = \frac{1}{1 + \left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \tag{58}$$

$$n_{\nu}(x) = \frac{\nu\nu_0(1-x)}{\nu\nu_0(1-x) + (1-\nu)(1-\nu_0)x} \tag{59}$$

$$n_{\nu_*}(x) = \frac{\nu_*^2(1-x)}{\nu_*^2(1-x) + (1-\nu_*)^2 x} \tag{60}$$

If $\gamma = 1$, then $o_{\gamma} = c$, if $\gamma = -1$, then $o_{\gamma} = d$, if $\nu_0 = \frac{1}{2}$

$$n_{\nu}(x) = \frac{\nu(1-x)}{\nu(1-x) + (1-\nu)x} \tag{61}$$

if $\nu_* = \frac{1}{2}$

$$n_{\frac{1}{2}*}(x) = 1 - x \tag{62}$$

VIII. Conclusion

We show, that the necessary and sufficient condition of DeMorgan class of generalized operators (14), (15) is (17), i.e.:

$$f_c^{-1}(x) = n f_d^{-1}(ax) \qquad a \neq 0$$

We show, that the negation (25)

$$n(x) = f_d^{-1}(a f_c(x)) \qquad a \neq 0$$

is involutive iff there is $k(x)$, so that (26)

$$f_c(x) = \frac{1}{a} k(f_d(x))$$

is valid and

$$k^{-1}(x) = k(x)$$

We give the general form of the negation (34)

$$n(x) = f^{-1}(k(f(x)))$$

We show, that all negation can be given in this form (theorem 7).

After introducing the reciprocal system, we find, that the conjunctive and disjunctive operator can be written in a common form

$$o_\gamma(x,y) = f^{-1}(f^\gamma(x) + f^\gamma(y))^{\frac{1}{\gamma}}$$

and in fuzzy logic mostly need min and max operator can be get as a limes of γ.

We give the general form of Dombi operator, which serves for the basic connectives of pliant logic

$$o_\gamma(x_1, x_2, ..., x_n) = \frac{1}{1 + \left(\sum_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^\gamma\right)^{\frac{1}{\gamma}}}$$

$$n_\nu(x) = \frac{\nu\nu_0(1-x)}{\nu\nu_0(1-x) + (1-\nu)(1-\nu_0)x}$$

IX. Appendix

Theorem 17: $n(x)$ is negation iff

$$n(x) = f_*^{-1}(n_0(f_*(x))) \tag{63}$$

where $f_*(x)$ is the generator function of negation and $f_* : [0,1] \longrightarrow [0,1]$ is a strictly monotonously continuous function, on which $f_*(0) = 0$ and $f_*(1) = 1$ and $n_0(x)$ is negation.

Proof: I. (Sufficient) $n(x) = f_*^{-1}(n_0(f_*(x)))$ is negation, because:

1. the negation is continuous, because the composition of continuous functions is also continuous.

2. it is strictly monotonously decreasing, because if $x_1 < x_2$, then $f_*(x_1) < f_*(x_2)$, so $1 - f_*(x_1) > 1 - f_*(x_2)$, using the strict monotonicity of $f_*(x)$ again $f_*^{-1}(1 - f_*(x_1)) > f_*^{-1}(1 - f_*(x_2))$.

3. the negation is involutive, because

$$\begin{aligned} n(x) &= f_*^{-1}(n_0(f_*(n(x)))) \\ &= f_*^{-1}\left(n_0 f_*\left(f_*^{-1} n_0(f_*(x))\right)\right) \\ &= f_*^{-1}(n_0(n_0(f_*(x)))) = f_*^{-1}(f_*(x)) = x \end{aligned}$$

II. (Necessity) Let's divide $[0,1]$ intervallum into two part with the help of ν, ν_0.and let $N = \{x \mid x \leq \nu_0\}$ and $P = \{x \mid x > \nu_0\}$, where $\nu = n(\nu)$. Let $N' = \{x \mid x \leq \nu\}$ and $P' = \{x \mid x > \nu\}$, where $\nu_0 = n_0(\nu_0)$. Let $n(x)$ and $n_0(x)$ be negation,
Define $h : N \longrightarrow N'$ strictly monotonously increasing function and

$$f_*(x) = \begin{cases} h(x) & if \quad x \in N \\ n_0(h(n(x))) & if \quad x \in P \end{cases}$$

1. $f_* : [0,1] \longrightarrow [0,1]$, because if $x \in N$, then $h(x) = f_*(x) \in N'$, and if $x \in P$, then $h(n(x)) \in N'$ and so $n_0(h(n(x))) \in P'$.

2. f_* monotonously increasing, because in the case of $x < y$ $(x, y \in [0,1])$:

(a) if $x, y \in N$, then $f_*(x) = h(x)$ and h monotonously increasing, so $f_*(x) < f_*(y)$

(b) if $x \in N$ and $y \in P$, then $f_*(x) = h(x) \in N'$.
Because $y \in P$, that's why $n(y) \in N$, so $h(n(y)) \in N'$ and that's why $n_0(h(n(y))) \in P'$, therefore $f_*(y) \in P'$, so $f_*(x) < f_*(y)$;

(c) if $x, y \in P$ and $x < y$, then $n(x) > n(y)$, so $h(n(x)) > h(n(y))$

3. f_* continuous, because

(a) if $x < \nu$, then $f_*(x) = h(x)$ is continuous

(b) if $x > \nu$, then $f_*(x) = n_0(h(n(y)))$, i.e. it will be the composition of continuous functions, so it is continuous itself too.

(c) if $x = \nu$, then to the continuousity of f_* we have to confirm $h(\nu) = n_0(h(n(\nu)))$ equality $h(\nu) = \nu'$
$n_0(h(n(\nu))) = n_0(h(\nu)) = n_0(\nu') = \nu$

4. $f_*(0) = 0$, because $h(0) = 0$
$f_*(1) = 1$ because
$f_*(1) = n_0(h(n(1))) = n_0(h(0)) = n_0(0) = 1$

5. We shall show, that
$n(x) = f_*^{-1}(n_0(f_*(x)))$. f_* On the basis of definition it's easy to see, that

$$f_*^{-1}(x) = \begin{cases} h^{-1}(x) & if \quad x \in N' \\ n_v\left(h^{-1}(n_0(x))\right) & if \quad x \in P' \end{cases}$$

(a) If $x \in N$, then

$$\begin{aligned} f_*^{-1}(n_0(f_*(x))) &= f_*^{-1}(n_0(h(x))) = \\ &= n\left(h^{-1}(n_0(n_0(h(x))))\right) = \\ &= n\left(h^{-1}(h(x))\right) = n(x) \end{aligned}$$

(b) If $x \in P$, then

$$\begin{aligned} f_*^{-1}\left(n_0\left(f_*\left(x\right)\right)\right) &= f_*^{-1}\left(n_0\left(n_0\left(h\left(n\left(x\right)\right)\right)\right)\right) = \\ &= f_*^{-1}\left(h\left(n\left(x\right)\right)\right) = \\ &= f_*^{-1}\left(f_*\left(n\left(x\right)\right)\right) = n\left(x\right) \end{aligned}$$

■

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