Pliant system

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Abstract: Fuzzy theory is very popular and useful on the control field. The main problem arises when we ask for how can be chosen the membership function and operators. There are some consensus, although theoretical basis of the problem is missing. Our objectives is to give an answer for the assessment of membership function, of logical operators and of unary operators.

I. INTRODUCTION

1.1. Critical remarks

Fuzzy theory has met interest in applications over the past 20 years. On one hand there are a number of reasons for the popularity of the word “fuzzy”, on the other hand it is not clear what really fuzzy means and we can recognized also that the critical view of the fuzzy also not decrease. It is hard to find a congress on Artificial Intelligence with a fuzzy section, although fuzzy is for modeling human concepts and using computers the fuzzy algorithms runs on it well. Because of this out of consideration now fuzzy belongs to Computational Intelligence. Books on fuzzy theory have offered many divergent concepts developed since years, but no one can find summary of the theoretical basis of it. Fuzzy has many faces. Some directions are well established. One can find good theoretical results and successful applications are everywhere. Fuzzy needs more theoretical investigation and researchers have to find now a sound system. The state of the fuzzy world seems to be similar to the state of the probability theory before Kolmogorov had written his famous book on the basic term of probability theory. Maybe fuzzy theory has reached its own shadow line. One of the leading researchers Dubois and Prade also recognized this situation in their manifesto [4].

In this article we do not want to changed the fuzzy world and not want to solve all the problems (it is too large, too many people works on it, min and max operators in certain area works well) although we offer a special calculus with consistent operator system which could be answers for some problems of the fuzzy concept.

1.2. Zadeh’s fuzzy concept

Lets go back to the original article of Zadeh. He introduced two basic terms: fuzzy set operators (min, max product etc.) and the membership function which is the extension of the classical characteristic function. The famous original example was the word “young”. Both of them causes a lot of confusion. Inquire people on a fuzzy congress about how they asses and understand membership function, we get dozen of answers and considering the operators we only have to mention more then 30 different kind from it exist now.
Besides intersection and union the inclusion (in fuzzy logic implication) also plays an important role (fuzzy control). Here the situation is similar to the case of connectives. Going further, we can mention the defuzzification method too, where also to much solution exist. Summarizing the above mentioned, everywhere are too much alternatives, freedom is high and this does not help for the unification effort, to get a good system.
What did want to introduced Zadeh the fuzzy sets for? To support the human decision making, but there are only some articles dealing with multicriteria decision making. In this article our main objectives to go back to this topic.
To carry out successful a decision making process we have to deal with values, weights, threshold values, decision value, logical operators, aggregation and preference relation. Values are everywhere, important is to define a expectation levels and normalizing the range of

* This work was partially supported by OTKA T20150 and MKM 220.

the values (i.e. find membership function), so that the different expectation levels transformed into one common value (to constitute a common basis) called threshold (or decision) level.

Starting with the Dombi operators we are given a consistent system, where all operators (conjunction, disjunction, negation, aggregation, sharpening operator and hedges) have only one generator function. We handle five different values:

- evaluation of the object
- evaluation of the criteria (i.e. the importance factor or weights),
- decision level
- thresholds
- sharpness.

The operators together with particular membership function (serving to describe a the evaluative preference) are called evaluation operators. We have a lot of other results based on this concept, which gives the reason why we distinguish this approach from the fuzzy theory. Let us mention some of it, but because of the shortage of the place we only give an overview

To go from evaluation to the appraisal we have to use a modus ponens like inference. In pliant logic we do it by introducing the appraisal process. Compare with fuzzy theory where approximate reasoning plays cardinal role and the logic has only one level, here we distinguish the object level from the meta level. On the object level we have inequalities (evaluative preferences) and on the meta level we use the new operators. Inference is in this case a simple optimization process.

Other approach is using evaluative operators on the object level and as in fuzzy theory, we build the weighted averages of the rules on meta level. The advantages here, that the surface of the rules very easy yielding, because all parameters have meaning and with the sharpness can change the distance from the classical sharp logic.

Now with the help of pliant logic is very easy to translate the rules into mathematical expression. The system identification can be done by neural network as well as by other optimization technique.

We show that the fuzzy concept naturally correspond the neural approach, i.e. the aggregation using evaluative preference is the same as in neural computation the gate function.

II MEMBERSHIPFUNCTION

2.1. Zadeh's view

Since Zadeh introduced the notation of fuzzy sets one of the main difficulties has been with the meaning and assess the membership function. It is hard to find good solution , because the membership function is context dependent, it has a subjective nature, from the point of view of measurement theory a lot of condition should satisfy, empirically hard to verify etc. Let us analyze the example of the word "young". Young depends only on one variable (although it depends on several, not only on ages). The membership function should be close to the characteristic function. Better to say it is a continuously approximation of the characteristic function. In this case the characteristic function is an inequality relation: to the class of young's belong all of people who's age is less then 25. (or people who's birth date is not earlier then 1972).

2.2. Decision- and threshold value

The year 25 is a decision value (level). The membership function in this case is an S-shape function, and the function value at 25 is 0.5. The value 0.5 is an universal threshold value. If we want to make decision on the basis of the function value, then we have to compare with the this value and if it is less, than our decision is not young otherwise is young. The threshold value can be changed and with the help of an unary operator (see chapter...) we can change this value.

The result of the comparison (it is distance like measure) indicate "how far" is a given people from the class of young people (if he/she not belongs to this class.) using the term of "not young" representing it with a similar function we can measure how far he/she from the not young people (i.e. how young he/she).

2.3. Fuziness measure, sharpness

The above mentioned measure essentially is not an other then the approximation accuracy of the characteristic function. The approximation can be characterized by the slope of the function at the decision value (the first deviate of the function at 25). In fuzzy theory we call it fuzziness, or sharpness and this value from case to case changes. For better controlling it we will introduce the sharpness operator. To go back to the word young on the x axis was the year, modeling other concept the meaning of the axes changes. It can be probability, as utility, time etc. in our approach we do not care what is on the x axis, how can it be asses, so we avoid the a lot of problems which concern other area of science, as psychology, sociology, measurement theory, human utility, risk perceptions. It is impossible to constitute theoretical framework on empirical evidence.

2.4. Evaluation relation

Summarizing the above mentioned considerations we give the following definition:
Definition: Evaluation relation is a function $\sigma(x)$ equipped with the following parameters: decision value (or level) a, sharpness λ, universal threshold value ν^0 which is usually *0.5* and

$$\lim_{x\to-\infty}\sigma_a^\lambda(x)=0,$$

$$\lim_{x\to\infty}\sigma_a^\lambda(x)=1,$$

$$\lim_{\lambda\to\infty}\sigma_a^\lambda(x)=\begin{cases}0 & if\ x<a\\ \nu^0 & if\ x=a\\ 1 & if\ x>a\end{cases}$$

We show that satisfying some reasonable axioms the function has the following form:

$$\sigma_a^\lambda(x)=\frac{1}{1+e^{-\frac{\lambda}{\nu(1-\nu)}(x-a)}}$$

In fuzzy theory membership function usually represents an interval (middle ages). It has a definite upper and lower bound. With evaluation relations we can reconstruct it using set operator (intersection). Approximating with evaluation relations young- old people (threshold values are 25 and 55) one can get the middle ages people using the negation- and the intersection operators.
The most important step was here not to take into consideration the semantically meaning of the x axis, i.e. what over we define the evaluation relation.

We do not loose to much, because a membership function approximable with evaluation relations using a well established *l(x)* transformation on x,

$$|\mu(x)-\sigma(l(x))|<\varepsilon$$

l(x) is responsible for all other features of the membership function and so the evaluation theory do not want to deal with task of other sciences.
All parameters of evaluation relations have a semantically meaning as threshold-, decision level and sharpness.

III. MOTIVATION AND MAIN IDEA

Zadeh in his first article introduces the min and max operators (and the product operator with its dual pairs) for intersection and union. During the years a lot of scientific investigations have been done to find good operators. Most important names in this field are Hamacher, Weber, Mizomuto, Dombi. We concentrate on the result of Dombi.
The so called Dombi operators has the following forms [1]:

$$c(x,y)=\frac{1}{1+\left(\left(\frac{1-x}{x}\right)^\lambda+\left(\frac{1-y}{y}\right)^\lambda\right)^{\frac{1}{\lambda}}}$$

$$d(x,y)=\frac{1}{1+\left(\left(\frac{1-x}{x}\right)^{-\lambda}+\left(\frac{1-y}{y}\right)^{-\lambda}\right)^{-\frac{1}{\lambda}}}$$

The most important fact is that the two operators has a common form. Only changing the sign of λ from one operator we can get the other. As a limit value we get the min and max operators.
Interesting result [2] is the rational form of the Dombi's aggregation operator:

$$a(x,y)=\frac{1}{1+\left(\frac{1-x}{x}\right)^u\left(\frac{1-y}{y}\right)^v}$$

where u and v the corresponding weights. In the same article one can find the special form of the negation:

$$n(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \left(\frac{x}{1-x}\right)}$$

where the ν value was interpreted as neutral value.

In the article [3] Dombi makes theoretical investigation to find good membership functions and the rational form of the function is:

$$\mu(x) = \frac{1}{1 + \left(\frac{\nu}{1-\nu}\right)^{\lambda-1} \left(\frac{1-x}{x}\right)^{\lambda}}$$

where λ is the sharpness and v is the decision level.

All above mentioned results formally are similar. On one hand in the next we will generalize and characterize this class of operators and on the other we will build up a consistent logical system equipped by unary operators. Until now in fuzzy theory there was not introduced theoretically well based unary operators.

It seems to be reasonable also only using such parameters as λ, ν and the task of the unary operators to change this values.

IV. GENERAL DEMORGAN CLASS AND DUAL SYSYTEM

There are a lot of operators in the field of fuzzy set. What is important from the application point of view? The operator should be behave well, so not only the continuity is important but the good analytical property, as the existence of the deviates (not only the first). Why? Because of the leering aspects usually is an optimization procedure and to carry out it successfully we need such kind of property. We want also that the operators on the neuron computing aspects behave well.

It is known if c(x,y) and d(x,y) strict monotonously increasing associative archimedian t-norm and t-conorm., then *c(x,y)* and *d(x,y)* has the following form:

$$c(x,y) = f^{-1}(f(x) + f(y))$$

$$d(x,y) = g^{-1}(g(x) + g(y))$$

where f and g are the corresponding generator function of the operators.

Let generalize the operators:

$$c(x,y) = f^{-1}(af(x) + bf(y))$$

$$d(x,y) = g^{-1}(ag(x) + bg(y))$$

It can be shown the three operator are in DeMorgan class if the following conditions hold:

$$n(x) = f^{-1}\left(\frac{f(\nu^0)}{g(\nu\ )} g(x)\right)$$

where for v^0 and v are valid

$$n(\nu) = \nu^0$$

The most interesting features of Dombi's operator was their common form. In this section we want to preserve this property for the general form of the operators. We define the dual system.

Definition: Let *f(x)* is the generator function of the conjunctive operator and *g(x)* the generator function of the disjunction operator. We call the two operator dual to each other if

$$f(x) = \frac{1}{g(x)}$$

It is easy to see that the definition is god and Dombi's operators are dual. The generator function is

$$f(x) = \left(\frac{1-x}{x}\right)^{\lambda}$$

V. NEGATION AND UNARY OPERATORS

From the result of section 4 it is easy to calculate the form of negation in a dual system .

$$n(x) = f^{-1}\left(\frac{f(\nu^0) f(\nu)}{f(x)}\right)$$

This negation is really a negation. The most important fact is that only using one generator function we can build a whole system, i.e. we can get the disjunction and negation too. In fuzzy theory there was no tight correspondence between the operators. Negation is a kind of transformation of the threshold value. If $v = v^0$ then

$$n(x) = f^{-1}\left(\frac{f^2(v^0)}{f(x)}\right)$$

Using Dombi's generator functions we get back the negation developed in the article [2]

In this section we will use the result of Dombi on membership function. Two type of operator will be examine here: one called hedges (changes the threshold value) and one responsible for the sharpness.

5.1. *Hedges*

Fuzzy sets is modeling for human expressions and an interface between the words and mathematical functions and operations. To go back to the word "young", one can be transformed with a hedge "very". The word "very young" is also an evaluation relation and differs only it's decision level. The new evaluation relation one can get changing the decision value or transformed the threshold value. (Shifting or Transformation). We examine the latter one.
One way is to go back Dombi's article on membership function. Denoting the hedges by $\tau(x)$ and let $\tau(x)$ continuously increasing with the following properties:

$$\tau_v(0) = 0$$
$$\tau_v(v) = v^0$$
$$\tau_v(1) = 1$$

From this comes

if $x < v$ then $\tau(x) < v^0$ and if
if $x > v$ then $\tau(x) > v^0$.

so v is now the new threshold.
Because of the general form of the membership is

$$\tau_v(x) = f^{-1}(af(x))$$

we can find a solution for a and the result is

$$\tau_v(x) = f^{-1}\left(f(v^0)\frac{f(x)}{f(v)}\right)$$

The other way to find a good transformation function building hedges is using negations with different threshold values. This approach has a semantically meaning too.

$$n_{v^0}(n_v(x)) = f^{-1}\left(f(v^0)\frac{f(x)}{f(v)}\right)$$

The other parameter of the evaluation relation is the sharpness.

5.2. *Sharpness operator*

In the theory of fuzzy sets always plays an important role of the sharpness. It was introduced the fuzziness measure measuring the sharpness. But only in the first period of the fuzzy research researchers were dealing with this. Because of the membership is an approach of the characteristic function it is reasonable to express it's efficiency. Let us denote the sharpness operator by $\chi^{(\lambda)}(x)$ and let $\chi^{(\lambda)}(x)$ continuously increasing with the following properties:

$$\chi^{(\lambda)}(0) = 0$$
$$\chi^{(\lambda)}(v^0) = v^0$$
$$\chi^{(\lambda)}(1) = 1$$
$$\left.\frac{d\chi^{(\lambda)}(x)}{dx}\right|_{v^0} = \lambda$$

It can be shown using Dombi's result on that

$$\chi^{(\lambda)}(x) = f^{-1}(f^{1-\lambda}(v^0) f^{\lambda}(x))$$

It can be seen that using the generator function of Dombi then

$$\sigma_a^{(\lambda)}(x) = \chi_{v^0}^{(\lambda)}(\sigma_a^{(1)}(x))$$

Three different operator were introduced negation, hedges, and the sharpness operator.
In the next section we deal with this three operator together.

5.3. *Modifiers: Common form of the unary operators*

The three operator has a common form:

$$\tau_\nu^{(\lambda)}(x) = f^{-1}\left(f(\nu^0)\frac{f^\lambda(x)}{f^\lambda(\nu)}\right)$$

If $\lambda = 1$ then we get the hedge,
If $\lambda = -1$ then we get the negation and denoted it by η
If $\nu = \nu^0$ then we get the sharpness operator.

In the following $\tau_\nu^{(\lambda)}(x)$ is called modifier operator.

5.4. *Independence property*

Until now we do not care with the generator function i.e. is the generator function of the conjunctive or disjunctive operator? It can be shown if the two operators are dual as we supposed, then the modifiers are the same and modifier operator is closed under composition and composition with evaluation relation.
Dombi's form of the modifier is

$$\tau_\nu^{(\lambda)}(x) = \frac{(1-\nu)^\lambda \nu^0 x^\lambda}{(1-\nu)^\lambda \nu^0 x^\lambda + \nu^\lambda (1-\nu^0)(1-x)^\lambda}$$

If ν^0 as usual is 0.5 then

$$\tau_\nu^{(\lambda)}(x) = \frac{(1-\nu)^\lambda x^\lambda}{(1-\nu)^\lambda x^\lambda + \nu^\lambda (1-x)^\lambda}$$

and if $\nu = \nu^0$ then we get Dombi's membership function [3].
The modifiers are also evalution relations. ν is the decision level, λ is the shrpness. The difference between the σ function and modifers is only the domain of the two functions.

VI. THERSHOLD AND LOGICAL OPERATORS

The main problem with the strict monotonous operator is that they are not idempotent. As in the modifiers the threshold will play an important role. We changes the operator system using on it unary operator.

Let us suppose that if all x_i are less then our ν^0 threshold level, it is reasonable to require that the result also has this properties and similar, if they are less then the threshold level the result also should be less. Our objectives to find an unary operator on conjunction and disjunction that this properties fulfills. The modifier in this case is

$$\tau(x) = f^{-1}(\tfrac{1}{n} f(x))$$

where n is the number of the arguments.
Using this modifiers to the operators we get

$$o(x_1, x_2, \ldots x_n) = f^{-1}(\tfrac{1}{n}\sum_{i=1}^{n} x_i)$$

where o is either conjunctive (ϕ) or disjunctive (ψ) operator. The operators do not loose the conjunctive and disjunctive characteristics with this modification and they are idempontent too. The form is equivalent with the mean operator. The specialty of this mean is that they behave conjunctive and disjunctive properties. In multicriteria decision making the weighting system is the most important question. In the next we summarized our results.

VII. WEIGHTS AND OPERATORS

Usually the problem to find weights in the decision process appears independent from the aggregation (operators used in the decision making). It can be shown that the weights and aggregation is closely related to each other. We show that the weighted aggregation is

$$o(x_1, w_1; x_1, w_1; \ldots; x_n, w_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right)$$

where the sum of the weights are 1.
Operators with weight are the generalization of the operators of the previous section of the mean operator.

VIII. AGGREGATION

In the article of Dombi [2] was examined the aggregation concept. It is the general solution of associative functional equation under particular conditions. We show that this kind of operator naturally relate to the dual operators. It is known if f a generator function of a logical operator (conjunctive or disjunctive) then f^{γ} generator function too.

$$o_{\gamma}(x_1, w_1; x_1, w_1; ...; x_n, w_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f^{\gamma}(x_i)\right)^{\frac{1}{\gamma}}$$

If $\gamma > 0$ then the operator is conjunctive
if $\gamma < 0$ then the operator is disjunctive
if γ goes to + infinity than the corresponding operator is min and
if γ goes to $-$infinity then the corresponding operator is max operator.
The main theory of this section is :
if $\gamma = 0$ then we get the weighted aggregation operator back i.e.

$$\theta(x_1, w_1; x_2, w_2; ....; x_n, w_n) = f^{-1}\left(\prod_{i=1}^{n} f^{w_i}(x_i)\right)$$

The parameter γ responsible for the type of the operator. The negation operator belonging to the aggregation operator also do not changes i.e. the same as derived at logical operators.

IX PLIANT LOGIC BASED ON DOMBI OPERATORS.

In the following we summarized the special realization of the evaluation concept using Dombi operators and evaluation relations. The logic with the notation system is:

values	$x, y, \quad x_i$
decision level	$a, b, \quad a_i$
threshold value	$\nu, \nu^{\circ}, \nu^{*}$
$n(\nu^{*}) = \nu^{*} \quad n(\nu) = \nu^{\circ}$	
importance, weight	u, v, w, w_i
sharpness	λ
type of the operator	γ

Notation in Pliant logic:

Unary operators

$\{a < x\}$	evaluation relation
$\{\nu \succ x\}$	negation
$\{x^{\sim}\}$	
$\{\nu \prec x\}$	modifiers
$\{x\}^{\lambda}$	sharpness
(w □ x)	weighting

Operators

$x\{\Delta\}^{\gamma}y$	conjunctiv
$x\{\nabla\}^{\gamma}y$	disjunctiv
$\{x\Diamond y\}^{\lambda}$	aggregation

We show only some expression as an example:

conjunction:

$$\{a_1 < x_1\}^{\lambda_1} \Delta \{a_2 < x_2\}^{\lambda_2} .... \Delta \{a_n < x_n\}^{\lambda_n} = \frac{1}{1 + \frac{1}{n}\sum_{i=1}^{n} e^{-\lambda_i (x_i - a_i)}}$$

disjunction:

$$\{a_1 < x_1\}^{\lambda_1} \nabla \{a_2 < x_2\}^{\lambda_2} .... \nabla \{a_n < x_n\}^{\lambda_n} = \frac{1}{1 + \left(\frac{1}{n}\sum_{i=1}^{n} e^{\lambda_i (x_i - a_i)}\right)^{-1}}$$

aggregation:

{(w$_1$ □ {a_1<x_1}) ◊ (w$_2$ □ {a_2<x_2})....

..... (w$_n$ □ {a_n<x_n}$^{\lambda}$)}$^{\lambda}$ =

$$= \frac{1}{1 + e^{-\lambda \sum_{i=1}^{n} w_i (x_i - a_i)}}$$

Aggregation is in pliant logic is not an other then neural approach.

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