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Universal characterization of non-transitive preferences

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Abstract

The purpose of this paper is to explore and characterize the non-transitivity of preferences in the Fishburn decision-making theory. We consider the case in which the decision outcomes are integers and the probability distributions on X are two-valued. The k-cyclicity of a preference is defined for any positive integer k, and it is shown that a k-cyclic preference exists for every k. We represent preferences with univariate functions and give one class of k-cyclic preference function. Finally, different preference functions are given in concave, convex and ε-linear form.

Key words: Utility theory; SSB utility theory; Non-transitivity; k-cyclicity

1. Introduction

The theories of preference comparisons under risk and under uncertainty have been widely adapted over the past 35 years. During this period there has been a growing awareness that human reasoned judgments often violate the basic assumptions of expected utility.

Violations of the axioms and the underlying principles of expected utility theory have been generated by certain experimental conditions and framing procedures. An important task for normative theory is to decide which violations of the von Neumann–Morgenstern axioms are experimental artifacts and which violations constitute fundamental rejections of the axioms by intelligent people. Many generalizations of the expected utility theories have been proposed. Systematic failures of the independence axiom or expectation principle have received special

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attention, but independence failures and intransitivities have not been ignored.

Fishburn (1982) introduced the 'non-transitive measurable utility' in which preferences are represented by the positive part of the skew-symmetric bilinear (SSB) functional on pairs of lotteries or risky decisions. On the other hand, the SSB utility theory preserves the continuity, convexity and monotonicity. Its representation for preference between risky prospects uses an SSB form ϕ on $P \times P$ (where P denote the convex set of probability distributions) rather than the bilinear form ϕ on P. This apparently modest generalization of linear utility has far-reaching consequences (Fishburn, 1984a, b).

In the first place, it accommodates phenomena (such as preference cycles, the preference reversal phenomenon and systematic failures of independence) that violate the axioms of linear theory and severely challenge its claim as a viable normative theory. Second, and despite the possibility of cyclic preferences, it supports a theory of choice by maximally preferred lotteries that guarantees such lotteries for all finitely based situations and ensures their independence from inferior infeasible alternatives.

Third, it adapts well to areas where linear utility has been applied. The SSB functional $\phi(x, y)$, measures the preference intensity between alternatives x, y and $\phi(x, y) > 0$ if and only if $x \succ y$. Fishburn supposed $\phi(x, y) = h(x - y)$, $x \geq y$, where $h(x - y)$ is a univariate real-valued function, such that the properties of the preference can be characterized with the properties of $h(x - y)$. Fishburn showed that with a particular choice of $h(x - y)$ we obtain cyclical preference. The question arises of how we can characterize the cyclicity of non-transitive preferences.

Fishburn (1984a) supposed also that the elements of P are two-valued. Such elements of P are typical examples of lotteries: to win \$$m$ with probability α and \$0 with probability $1 - \alpha$, denoted by $[m, \alpha]$. Throughout this paper, X will denote a non-empty set of potential decision outcomes and P will denote a convex set of probability distributions on X. Because the elements of X are the sums which we can win in the game, further let $X = \{0, 1, \ldots, n\}$ for any positive integer n. A preference cycle in SSB utility theory for $[m_1, p_1]$, $[m_2, p_2]$, $\ldots, [m_n, p_n]$ lotteries from P is: $[m_1, p_1] \succ [m_2, p_2] \succ \cdots \succ [m_n, p_n] \succ [m_1, p_1]$.

Fishburn gave an example with five lotteries, and presented a function $h(x - y)$ with which we obtain a preference cycle:

$$[6, 0.9] \succ [7, 0.8] \succ [8, 0.72] \succ [9, 0.66] \succ [10, 0.61] \succ [6, 0.9] .$$

In this paper we give a generalization of this cyclicity. We will define the k-cyclicity of any positive integer k. The preference is k-cyclic on the lotteries if

$$\begin{aligned}
&[m, p(m)] \succ [m+1, p(m+1)] \succ \cdots \succ [m+k, p(m+k)] \\
&\qquad \succ [m, p(m)] , \\
&[m, p(m)] \succ [m+1, p(m+1)] \succ \cdots \succ [m+2k, p(m+2k)] \\
&\qquad \succ [m, p(m)] , \\
&\vdots
\end{aligned}$$

$$[m, p(m)] > [m+1, p(m+1)] > \cdots > [m+lk, p(m+lk)]$$
$$> [m, p(m)], \quad \text{for } lk \le n-j.$$

Our main results: one class of k-cyclic preference function is given by the theory of finite difference equations, from the solution of the linear homogeneous second-order difference equation

$$g(m+2) - sg(m+1) + g(m) = 0,$$

for the special parameter s. In our case the explicit form of $h(x-y)$ is: $h(m) = v(m)g(m)$, where $g(m)$ is a solution of difference equation, positive on $\{1, \ldots, n\}$.

All solutions are for any non-zero real number $g(1)$ and for the parameter s, defined by $s := g(2)/g(1)$,

$$g(m) = \begin{cases} \dfrac{g(1)}{\sqrt{s^2-4}}\left(\left(\dfrac{s}{2} + \dfrac{\sqrt{s^2-4}}{2}\right)^m - \left(\dfrac{s}{2} - \dfrac{\sqrt{s^2-4}}{2}\right)^m\right), & \text{if } |s| > 2, \\ g(1)m, & \text{if } s = 2, \\ (-1)^m g(1)m, & \text{if } s = -2, \\ \dfrac{g(1)}{\sin\left(\arccos \dfrac{s}{2}\right)} \sin\left(m \arccos \dfrac{s}{2}\right), & \text{if } |s| < 2, \end{cases} \tag{1}$$

and for the function $v(m)\colon N \to R^+$ the following inequality system holds:

$$v(i) < v(m)v(m+i), \quad \text{if } i \ne rk,$$

and

$$v(pk) > v(m)v(m+rk), \quad \text{when } r \in N, r \le \left[\frac{n}{2k}\right],$$

where $j \le m$, $m+i$, $rk \le n$, for fixed positive integer k.

We give convex and concave k-cyclic preference functions. There is no linear preference function, but we can get a k-cyclic preference function which differs from the linear solution (2) by as little as we want. Therefore we introduce ε-linearity as the measure of the difference between $h(m)$ and $g(m)$. We show that an ε-linear k-cyclic solution exists. It is neither concave nor convex. Finally, examples are presented of k-cyclic preferences which have convex, concave and 0.4-linear form.

2. Preliminary definitions

Throughout, X will denote a non-empty set of potential decision outcomes, and P will denote a convex set of probability distributions on X. In this paper, $X =$

$\{0, 1, \ldots, n\}$ for positive integer n, and each $p \in P$ is a two-valued function from X into $[0, 1]$ that has $p(m) > 0$ for only one $m \in X$, that takes the value $1 - p(m)$ on 0 and for every $m \in X$ there exists only one $p \in P$ which has a positive value on m.

Definition 1. A set P of probability distributions on X is convex if $\lambda p + (1-\lambda)q$, which takes on the value $\lambda p(x) + (1-\lambda)q(x)$ for each $x \in X$, and is in P whenever $p, q \in P$ and $0 \le \lambda \le 1$.

Definition 2. A real-valued function ϕ on $X \times X$ is skew symmetric if

$$\phi(a, b) = -\phi(b, a), \quad \text{for all } a, b \in X.$$

Definition 3. Φ is a bilinear functional on $X \times X$ if it is real-valued and linear separately in each argument, such that

$$\phi(\lambda p + (1-\lambda)q, r) = \lambda\phi(p, r) + (1-\lambda)\phi(q, r)$$

and

$$\phi(p, \lambda q + (1-\lambda)r) = \lambda\phi(p, q) + (1-\lambda)\phi(p, r),$$

$$\text{for every } p, q, r \in P \text{ and } 0 \le \lambda \le 1.$$

Now consider the risky prospect $p \in P$ vs. outcome $y \in Y$. Since $\phi(x, y)$ measures the person's preference intensity for x over y for each x that has $p(x) > 0$, his expected intensity for p vs. y, written as $\phi(p, y)$, is given by the following:

Definition 4. The preference between $p \in P$ and $y \in X$ according to Fishburn (1984a, 1988) is

$$\phi(p, y) = \sum_{x \in X} p(x)\phi(x, y), \text{ so that } p > y \text{ if } \phi(p, y) > 0,\ p < y$$
$$\text{if } \phi(p, y) < 0, \text{ and } p \sim y \text{ if } \phi(p, y) = 0.$$

Finally, consider lottery p vs. lottery q. Since $\phi(p, y)$ is the person's expected intensity for p vs. y for each y that has $q(y) > 0$, his overall expected intensity for p vs. q, written as $\phi(p, q)$, is:

Definition 5. The preference between p and q if $p, q \in P$ is

$$\phi(p, q) = \sum_{x \in X} \sum_{x \in X} p(x)q(y)\phi(x, y).$$

We consider the case $\phi(x, y) = h(x - y)$, $x > y$, while $h(x - y)$ is a positive increasing real-valued function (Fishburn 1984a, 1988), called the preference function.

3. Example of preference intransitivity

Fishburn (1984a, 1988) gives an example of non-transitivity. When $[x, a]$ represents a lottery that pays \$$x$ with probability a and nothing otherwise, a number of people exhibit the cyclic pattern:

$$[6, 0.9] > [7, 0.8] > [8, 0.72] > [9, 0.66] > [10, 0.61] > [6, 0.9] .$$

In decision theory, this is known as a 4-cycle; see Kelsey (1984). Fishburn (1984a) shows how this preference cycle can be accounted for by SSB representation even when $\phi(x, y)$ depends only on the difference between x and y. If $h(x-y)$ is given by $h(1)=0.75$, $h(4)=4.6$, $h(6)=6.2$, $h(7)=6.8$, $h(8)=7.5$, $h(9)=7.8$, $h(10)=8$, then it can represent the preference cycle. From the definition:

$$\begin{aligned}\phi([6, 0.9], [7, 0.8]) &= (0.9)\cdot(0.8)\phi(6, 7) + (0.9)\cdot(1-0.8)\phi(6, 0) \\ &\quad + (1-0.9)\cdot(0.8)\phi(0, 7) \\ &= (-0.72)h(1) + (0.18)h(6) - (0.8)h(7) \\ &= (-0.72)\cdot(0.75) + (0.18)\cdot(6.2) - (0.08)\cdot(6.8) \\ &= 0.032 ,\end{aligned}$$

$$\phi([7, 0.8], [8, 0.72]) = 0.112,\ \phi([8, 0.72], [9, 0.66]) = 0.038 ,$$

$$\phi([9, 0.66], [10, 0.61]) = 0.046,\ \phi([10, 0.61], [6, 0.9]) = 0.837 .$$

4. Connection between preference and functional inequalities

The question arises: how can we characterize the cyclicity of non-transitive preferences? If $\phi(x, y) = h(x-y)$, then the properties of the preference relation (Fishburn, 1971; Burros, 1974) give conditions for the preference function $h(x-y)$. For example, if $h(x-y)$ is the preference function of a transitive preference, then:

$$\text{if } h(x-y) > 0 \text{ and } h(y-z) > 0, \text{ then } h(x-z) > 0,\ x, y, z \in X .$$

Therefore, the properties of the preference can be characterized with the properties of the function $h(x-y)$. Accordingly, we consider the function $h(x-y)$. First, we introduce the function $p(m)$. Since every $p \in P$ has a positive value on one and only one element of X and for every $m \in X$ there exists only one $p \in P$ which has a positive value on X, we can consider these positive values as the range of the univariate positive real-valued function $p(m)$. It is presumed that $p(m)$ is increasing. Now we introduce the k-cyclicity as the generalization of the cyclicity of the SSB utility theory.

Definition 7. Let $X = \{j, \ldots, n\}$ for positive integers j, n. The preference is then k-cyclic on X, for fixed $k \in N$, if

$$[m, p(m)] < [m+rk, p(m+rk)]$$

and

$$[m, p(m)] > [m+i, p(m+i)] , \quad \text{if } i \neq rk,$$

$$\text{for every } m, r, i \in N \text{ for which } j \leq m,\ m+rk,\ m+i \leq n,\ k < n-j .$$

For $k = 1$, this obviously gives the transitivity of the preference. In this paper the cyclicity of preference means the k-cyclicity of this preference for any positive integer k. The above question now aries in another form: Is there a positive, increasing, real-valued function $h(x - y)$ such that the preference is k-cyclic on the interval $[j, n]$? The condition of k-cyclicity with functional $\phi(x, y)$ is

$$\phi([m, p(m)], [m + i, p(m + i)]) > 0 , \quad \text{if } i \neq rk ,$$

$$\phi([m, p(m)], [m + i, p(m + i)]) < 0 , \quad \text{if } i = rk,\ j < i, r, k, m < n .$$

According to Definition 5:

$$p(m)p(m + i)\phi(m, m + i) + p(m)(1 - p(m + i))\phi(m, 0) + (1 - p(m))p(m + i)\phi(0, m + i) > 0 , \quad \text{if } i \neq rk ,$$

$$p(m)p(m + i)\phi(m, m + i) + p(m)(1 - p(m + i))\phi(m, 0) + (1 - p(m))p(m + i)\phi(0, m + i) < 0 , \quad \text{if } i = rk .$$

Since $p(m)$ is positive, we can divide by $p(m)p(m + i)$:

$$-h(i) + \frac{1 - p(m + i)}{p(m + i)}\, h(m) - \frac{1 - p(m)}{p(m)}\, h(m + i) > 0 , \quad \text{if } i \neq rk ,$$

$$-h(i) + \frac{1 - p(m + i)}{p(m + i)}\, h(m) - \frac{1 - p(m)}{p(m)}\, h(m + i) < 0 , \quad \text{if } i = rk .$$

Let us denote $(1 - p(m))/p(m)$ by $f(m)$:

$$f(m + i)h(m) - h(m + i)f(m) > h(i) , \quad \text{if } i \neq rk ,$$

$$f(m + i)h(m) - h(m + i)f(m) < h(i) , \quad \text{if } i = rk,\ j \leq m, m + i, rk \leq n . \tag{2}$$

This is a functional inequality system. It can be seen that the existence of the positive real-valued increasing solutions $h(m)$ and $f(m)$ of (2) on the interval $[j, n]$ is equivalent to the existence of k-cyclicity.

5. Characterization of non-transitivity with preference functions

Lemma 1. *We can get a solution $h(m)$ of (2) in the form $h(m) = v(m)g(m)$, where $g(m)$ is the solution of the functional equation*

$$F(m + i)g(m) - g(m + i)F(m) = g(i), \text{ for } F(m) = \frac{f(m)}{v(m)}$$

$$\text{for } 1 \leq m, i, m + i \leq n , \text{ and } g(m) \text{ is positive on } \{1, \ldots, n\} , \tag{3}$$

and $v(m)$ is positive and satisfies:

$$v(i) < v(m)v(m + i) , \quad \text{if } i \neq rk ,$$

and

$$v(rk) > v(m)v(m+rk), \text{ where } r \in N, r \leq \left[\frac{n}{2k}\right],$$

$$\text{where } j \leq m, m+i, rk \leq n. \tag{4}$$

Proof. Let $v(m)$ be as in (4), (see Lemma 2) and let $F(m) = f(m)/v(m)$. To solve (3) let $u(m) = F(m)/g(m)$. The function $u(m)$ exists, because $g(m) > 0$ for every $m \in \{1, \ldots, n\}$. Substituting $F(m)$ by $u(m)g(m)$ in the form (3):

$$g(m)g(m+i)[u(m+i) - u(m)] = g(i).$$

Since $g(m) > 0$ for every $m \in \{1, \ldots, n\}$:

$$u(m+i) - u(m) = \frac{g(i)}{g(m)g(m+i)}, \quad \text{for } 1 \leq m, i, m+i \leq n. \tag{5}$$

Then multiply the right-hand side of (5) by $v(i)/[v(m)v(m+i)]$:

$$u(m+i) - u(m) > \frac{g(i)}{g(m)g(m+i)} \frac{v(i)}{v(m)v(m+i)}, \quad \text{if } i \neq rk,$$

$$u(m+i) - u(m) < \frac{g(i)}{g(m)g(m+i)} \frac{v(i)}{v(m)v(m+i)}, \quad \text{if } i = rk,$$

$$\text{where } r \in N, r \leq \left[\frac{n}{2k}\right], j \leq m, m+i, rk \leq n.$$

Multiply by $g(m)g(m+i)v(m)v(m+i)$:

$$v(m+i)u(m+i)g(m+i)v(m)g(m) - v(m)u(m)g(m)v(m+i)g(m+i) > v(i)g(i), \quad \text{if } i = rk,$$

$$v(m+i)u(m+i)g(m+i)v(m)g(m) - v(m)u(m)g(m)v(m+i)g(m+i) < v(i)g(i),$$

$$\text{if } i = rk, \text{ where } r \in N, r \leq \left[\frac{n}{2k}\right], j \leq m, m+i, rk \leq n.$$

Using $F(m) = u(m)g(m)$, $f(m) = v(m)F(m)$ and $h(m) = v(m)g(m)$ we get (2):

$$f(m+i)h(m) - h(m+i)f(m) > h(i), \quad \text{if } i \neq rk,$$

$$f(m+i)h(m) - h(m+i)f(m) < h(i), \quad \text{if } i = rk, j \leq m, m+i, rk \leq n.$$

□

Lemma 2. *There exists a function* $v(m): N \to R^+$ *such that, for every* $k, n \in N$, *if* $2k \leq n$, *there exists* $j \in N$ *such that* (4) *holds on the interval* $[j, n]$.

Proof. Let us take the following part of inequality system (4):

(a) $v(rk) > v(rk)v(2rk)$,

(b) $v(2rk - h) < v(h)v(2rk)$, when $h \neq rk$,

(c) $v(h) < v(2rk - h)v(2rk)$,

for any $r, k, h \in N$ when $1 \leq 2rk, h \leq n$. Then, from (a) we get $v(2rk) < 1$, while from (b) and (c), after multiplication of (c) by $v(2rk)$:

$$v(2rk - h) < v(h)v(2rk) < v(2rk - h)v^2(2rk) .$$

Since $v(m)$ is positive, $v(2rk) > 1$ is in contradiction with $v(2rk) < 1$. Accordingly, let $l = \max\{r \mid 2rk \in n\}$ and then let $j = lk + 1$. It is easy to see that this is a minimal such value that system (4) does not have the above-mentioned contradiction on $[j, n]$, and we have a function $v(m)$ which satisfies the inequality system (4) on $[j, n]$. Let $v(k) = \cdots = v(rk) = d$, $v[(l + 1)k] = q$ and $v(i) = t$ for all other i, where $d, q, t \in R^+$ and d, q, t are such that $d > tq$ and $q > t$. This $v(m)$ gives the solution of inequality system (4). □

To solve (3) in Lemma 1, we get $F(m)$ in the form $F(m) = u(m)g(m)$, so

$$u(m + i) - u(m) = \frac{g(i)}{g(m)g(m + i)} , \quad \text{for } 1 \leq m, i, m + 1 \leq n . \tag{5}$$

And

$$\begin{aligned} u(m + i) - u(m) = {} & [u(m + i) - u(m + i - 1)] \\ & + [u(m + i - 1) - u(m + i - 2)] + \cdots \\ & + [u(m + 1) - u(m)] , \end{aligned}$$

then $g(m)$ is a solution of (5) if and only if it is a solution of

$$\frac{g(1)}{g(m)g(m + 1)} + \cdots + \frac{g(1)}{g(m + i - 1)g(m + i)} = \frac{g(i)}{g(m)g(m + i)} , \tag{6}$$

for $1 \leq m, i, m + i \leq n$. Hence $g(m)$ is a solution of (3) if and only if $g(m)$ is a solution of (6).

Lemma 3. *Let n be a positive integer with $n > 1$ and let the function $g(m)$: $\{1, \ldots, n\} \to R$ be such that $g(m) \neq 0$ for every $m \in \{1, \ldots, n\}$. If (6) holds for $i = 2$ and for every m for which $1 \leq m \leq n - 2$, then (3) holds for every i, m where $1 \leq m, i, m + i \leq n$.*

Proof. We shall prove that from (6):

$$\frac{g(1)}{g(m)g(m + 1)} + \cdots + \frac{g(1)}{g(m + i)g(m + i + 1)} = \frac{g(i + 1)}{g(m)g(m + i + 1)} . \tag{7}$$

Adding $g(1)/[g(m + i)g(m + i + 1)]$ to (6). So on the left-hand side of (6), we have the left-hand side of (7). Accordingly we must prove that the right-hand sides are equivalent, i.e.

$$\frac{g(i)}{g(m)g(m + i)} + \frac{g(1)}{g(m + i)g(m + i + 1)} = \frac{g(i + 1)}{g(m)g(m + i + 1)} .$$

This is equivalent to

$$g(1)g(m) = g(i + 1)g(i + m) - g(i)g(i + m + 1) . \tag{8}$$

We suppose that (6) holds for $i=2$ and for $1 \le m \le n-2$, i.e.

$$\frac{g(1)}{g(m)g(m+1)} + \frac{g(1)}{g(m+1)g(m+2)} = \frac{g(2)}{g(m)g(m+2)}\,,$$

which is equivalent to

$$g(1)g(m) = g(2)g(m+1) - g(1)g(m+2)\,, \tag{9}$$

i.e.

$$g(m+2) = sg(m+1) - g(m)\,, \quad \text{for } s := \frac{g(2)}{g(1)}\,. \tag{10}$$

Adding $sg(2)g(m+2) - sg(2)g(m+2)$ to both sides in (9):

$$\begin{aligned} g(1)g(m) = {} & sg(2)g(m+2) - g(1)g(m+2) \\ & - [sg(2)g(m+2) - g(2)g(m+1)]\,. \end{aligned} \tag{11}$$

Using (10):

$$\begin{aligned} sg(2)g(m+2) - g(1)g(m+2) &= [sg(2) - g(1)]g(m+2) \\ &= g(3)g(m+2) \end{aligned}$$

and

$$\begin{aligned} sg(2)g(m+2) - g(2)g(m+1) &= g(2)[sg(m+2) - g(m+1)] \\ &= g(2)g(m+3)\,. \end{aligned}$$

Substituting this into (11), we get

$$g(1)g(m) = g(3)g(m+2) - g(2)g(m+3)\,. \tag{12}$$

Thus, every argument on the right-hand side of (9) is increased by 1 in (12). Equations (9) and (12) are special cases of the form

$$\begin{aligned} & g(1)g(m) = g(k)g(k+m-1) - g(k-1)g(k+m)\,, \\ & \quad k \in \{1, \ldots, n-1\}\,, \end{aligned} \tag{13}$$

for $k=1$ and for $k=2$. Adding $sg(k)g(k+m) - sg(k)g(k+m)$ to the right-hand side of (13), we get

$$\begin{aligned} g(1)g(m) = {} & sg(k)g(k+m) - g(k-1)g(k+m) \\ & - [sg(k)g(k+m) - g(k)g(k+m-1)]\,. \end{aligned} \tag{14}$$

Using (10):

$$\begin{aligned} & sg(k)g(k+m) - g(k-1)g(k+m) \\ & \quad = [sg(k) - g(k-1)]g(k+m) = g(k+1)g(k+m) \end{aligned}$$

and

$$sg(k)g(k+m) - g(k)g(k+m-1) = g(k)g(k+m+1)\,.$$

Substituting this result into (14), we get

$$g(1)g(m) = g(k+1)g(k+m) - g(k)g(k+m+1) .$$

Here, every argument on the right-hand side is increased by 1 relative to (13). In $i-1$ steps from (9) we get (8):

$$g(1)g(m) = g(i+1)g(i+m) - g(i)g(i+m+1) .$$

This means that if (6) holds for $i=2$ and for $m \in \{1, \ldots, n-2\}$, then (6) holds for every i, m for which $1 \le m,\ i,\ i+m \le n$. Thus, the solutions of (6) and (10) are the same functions. □

Lemma 4. *Let n be a positive integer with $n > 1$ and let the functions $f(m)$, $g(m)$: $\{1, \ldots, n\} \to R$ be such that $g(m) \neq 0$ for every $m \in \{1, \ldots, n\}$. Then, $f(m)$ and $g(m)$ are solutions of* (3) *if and only if $g(m)$ is as in* (2).

Proof. From Lemma 3 we get that the solutions of (6) are the same as the solutions of (10). In another form, (10) is

$$g(m+2) - sg(m+1) + g(m) = 0 , \tag{15}$$

which is a second-order homogeneous linear difference equation. The solutions of (15) with the parameter s are as in (1); see Miller (1960) and Bahvalov (1977). □

Stability of the solutions. The theory of finite difference equations gives the following theorem:

Theorem (Stoer and Bulirsch, 1980). *The stability of the solution $g(m)$ of the homogeneous linear difference equation,*

$$g(n+j) + a_{j-1}g(n+j-1) + \cdots + a_0 g(n) = 0 ,$$

is equivalent to the condition that the roots of the characteristic equation,

$$Q(z) = z^j + a_{j-1}z^{j-1} + \cdots + a_0 ,$$

are in the unit circle and there are no multiple roots on the line of the unit circle.

Therefore, if $|s| > 2$, then the solution is not stable because, if $s > 2$, then $(s/2) + \sqrt{s^2-4}/2 > 1$, and if $s < -2$, then $(s/2) - \sqrt{s^2-4}/2 < -1$, and thus these roots are not in the unit circle. If $|s| = 2$, then the root is on the line of the unit circle and is a 2-fold root. In the case $|s| < 2$, both roots are on the line of the unit circle and are not multiple roots. Therefore, the solution is stable.

In our theorem we give one class of the k-cyclic preference functions and we show convex and concave preference functions for each k. No linear preference function exists. However, we can obtain a k-cyclic preference function which differs from the linear solution $g(m)$ of (3) as we want. Therefore, we introduce ε-linearity as the measure of the difference between $h(m)$ and $g(m)$ and generally between an arbitrary function $c(m)$ and the linear function $am + b$.

Definition 8. Let ε be any positive real number. The function $c(m)$: $\{1, \ldots, n\} \to R$ is ε-linear if there exists a linear function $am + b$ such that $|h(m) - (am + b)| < \varepsilon$ holds for every m.

Our main results may be summarized up in the following theorem:

Theorem. *For every $k, n \in N$, where $2k \leq n$, and for every $\varepsilon > 0$, there exists $j \in N$, such that the preference $>$ is k-cyclic on the interval $[j, n]$ and there exist k-cyclic preference function $h(m)$ for every k in the form $h(m) = v(m)g(m)$, where $g(m)$ is the solution of* (3) *so that $g(m)$ is positive on $\{1, \ldots, n\}$ and $v(m)$ is a solution of* (4) *and $h(m)$ may be convex, concave and ε-linear.*

Proof. The theorem states that there exist strictly monotone increasing positive solutions of the functional inequality system (2):

$$f(m+i)h(m) - h(m+i)f(m) > h(i)\,, \quad \text{if } i \neq rk\,,$$

and

$$f(m+i)h(m) - h(m+i)f(m) < h(i)\,, \quad \text{if } i = rk\,,$$

for every k, in the form $h(m) = v(m)g(m)$, where $g(m)$, $v(m)$ as in theorem, and $h(m)$ may be convex, concave and ε-linear. We can get the form of $h(m)$ from Lemma 1. Lemma 3 gives the solution of the second-order homogeneous linear difference equation (15). We seek $h(m)$ in concave, convex and ε-linear form.

If $s > 2$, and $g(1) > 0$, then $g(m)$ is positive and monotone increasing. Let now $\varepsilon_0 = \min\{g(m) - g(m-1) \,|\, 2 \leq m \leq n\}$ an let t, q, d be as given in Lemma 2. It is easy to see that if $s(m)$ is a discrete convex (concave) function, there exists a positive d, for which, if $|w(m) - s(m)| < d$ for any function $w(m)$, then $w(m)$ is convex (concave). Let $\nu < \min\{\varepsilon_0, d\}$ and $1 < t, q, d \leq 1 + \nu$, and let j be as given in Lemma 2. Hence, on the interval $[j, n]$, $h(m) = v(m)g(m)$ for $s > 2$ will be a strictly monotone increasing positive convex function.

For $s = 2$ let $b = \min\{\varepsilon / g(m) \,|\, 1 \leq m \leq n\}$ and let $\nu < \min\{\varepsilon_0, b\}$ and let t, q, d as in Lemma 2. Then, $|h(m) - g(m)| = |v(m)g(m) - g(m)| < |(1 + \nu)g(m) - g(m)| = g(m) < \varepsilon$, so $h(m)$ is the ε-linear solution of (2). If $0 < s < 2$, then the period of the solution depends on the parameter s. Thus the magnitude of the period will be changed for any $n \in N$. Accordingly, we can take a monotonously increasing positive quarter-period of $g(m)$ on $[1, n]$. Let n, t, q, d, j be as in Lemma 2 and Lemma 3, then $h(m) = v(m)g(m)$ gives a strictly monotone increasing concave solution of (2). □

6. Example of a 3-cyclic preference function

We present an example of a 3-cyclic preference function on the interval [4, 11] with 0.4-linear function. Let the elements of a lottery by

$$[4, 0.24], [5, 0.19], [6, 0.16], [7, 0.14], [8, 0.12], [9, 0.11], [10, 0.1], [11, 0.09]\,.$$

Let the parameter $s = 2$ and let the parameters $d = 1.098$, $q = 1.065$ and $t = 1.03$. For the preference function, we have

$$h(1) = 1.03,\ h(2) = 2.06,\ h(3) = 3.29,\ h(4) = 4.12,$$
$$h(5) = 5.15,\ h(6) = 6.39,\ h(7) = 7.21,\ h(8) = 8.24,$$
$$h(9) = 9.27,\ h(10) = 10.3,\ h(11) = 11.33\ .$$

Hence, we have

$$\begin{aligned}\phi([4, 0.24], [5, 0.19]) &= -p(4)p(5)h(1) + p(4)(1 - p(5))h(4) \\ &\quad - (1 - p(4))p(5)h(5) \\ &= (-0.24)\cdot(0.19)\cdot(1.03) \\ &\quad + (0.24)\cdot(0.81)\cdot(4.12) \\ &\quad - (0.76)\cdot(0.19)\cdot(5.15) = 0.03\ ,\end{aligned}$$

which means that $[4, 0.24] > [5, 0.19]$. In the same way, we have

$$\phi([4, 0.24], [6, 0.16]) = 0.13,\ \phi([4, 0.24], [7, 0.14]) = -0.11\ ,$$
$$\phi([4, 0.24], [8, 0.12]) = 0.12,\ \phi([4, 0.24], [9, 0.11]) = 0.15\ ,$$
$$\phi([4, 0.24], [10, 0.1]) = -0.02,\ \phi([4, 0.24], [11, 0.09]) = 0.21\ ,$$

which shows the 3-cyclicity for $[4, 0.24]$. For $[5, 0.19]$, we have

$$\phi([5, 0.19], [6, 0.16]) = 0.06,\ \phi([5, 0.19], [7, 0.14]) = 0.06\ ,$$
$$\phi([5, 0.19], [8, 0.12]) = -0.11,\ \phi([5, 0.19], [9, 0.11]) = 0.12\ ,$$
$$\phi([5, 0.19], [10, 0.1]) = 0.15,\ \phi([5, 0.19], [11, 0.09]) = -0.02\ ,$$

which also shows the 3-cyclicity for $[5, 0.19]$. In this way, we find that the rule of 3-cyclicity holds for every lottery on the interval $[4, 11]$, which means that the preference relation is 3-cyclic on this interval. We can see this example in Fig. 1.

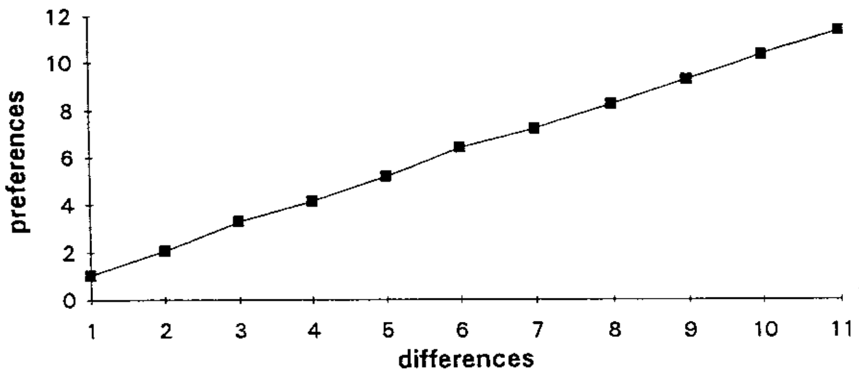

Fig. 1. 3-cyclic 0.4-linear preference function for $s = 2$.

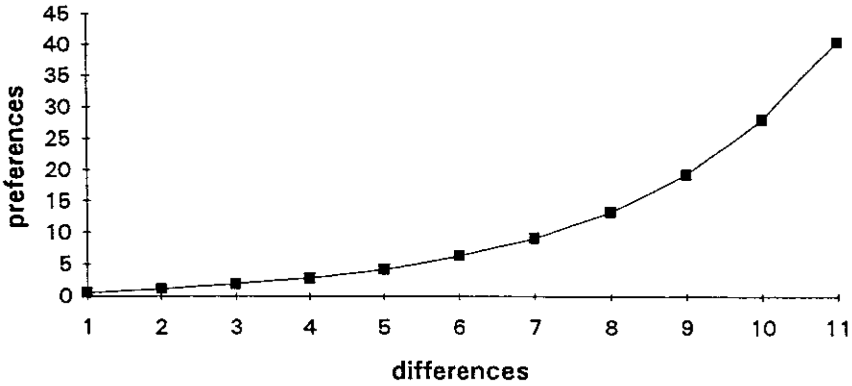

Fig. 2. 3-cyclic convex preference function for $s = 2.14$.

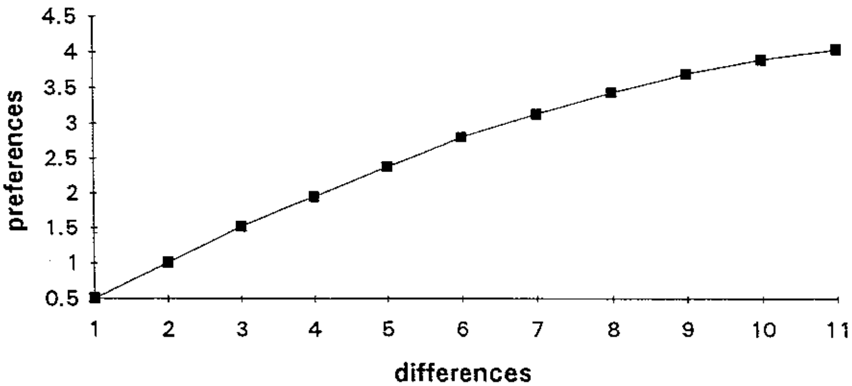

Fig. 3. 3-cyclic concave preference function for $s = 1.985$.

In Figs. 2 and 3 are two 3-cyclic preference functions: in case 1 a convex one, with parameter $s = 2.14$, and in case 2 a concave one, with parameter $s = 1.985$.

7. Summary and conclusions

It follows from the theory that for every positive integer k there exists a k-cyclic preference relation, and for every k we can characterize it with its preference function. We see that for fixed k the preference functions of k-cyclic preference relations may be given in several forms, from different classes of functions.

8. Open questions

We gave a multiplier function $v(m)$ for some $g(m)$. The first problem is: What is the form of $v(m)$ generally, near a given function $g(m)$? Let us derive another problem from this. Do we find a multiplier function $v(m)$ for every solution $g(m)$ of the mentioned finite difference equation? If these questions are answered, we can give a necessary and sufficient condition for the form of the preference function of the k-cyclic preference relation.

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