A General Framework for the Utility-Based and Outranking Methods

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Summary. In this paper on the utility and preference based decision process is presented in a unified way. We first deal with the utility based decision making comparing with the preference model. We define a universal preference function with useful properties and we show that the utility based decision making gives the same result as the preference based using the universal preference function. The preference based decision making process (the outranking approaches) can be get also with the help of the universal preference function i.e. we prove that there exists a unary function applying to the universal preference function getting the preference relations of the different outranking approaches.

1 Utility and Preference Based Decision Making

The most widespread methods used in multicriteria decision analysis are mainly of two types: the ones based on utility theory or on preference relations. Naturally, different methods provide different results even if the same information is used. The immediate question arises under which condition do the two approaches give the same result. More precisely, let a and b two actions (alternatives) and $c_1, c_2, \ldots, c_n$ different criteria, x_i and y_i are the utilities (evaluations) of a and b from the perspective of c_i, respectively.

Table 1.

	c_1	c_2	$\ldots$	c_n
a	x_1	x_2	$\ldots$	x_n
b	y_1	y_2	$\ldots$	y_n
	w_1	w_2	$\ldots$	w_n

Let $w_1, w_2, \ldots, w_n$ denote the nonnegative weights of the criteria such that they are normalized, i.e.:

$$\sum_{i=1}^{n} w_i = 1$$

The aggregated utility of actions a and b are

$$A = U(a) = \sum_{i=1}^{n} w_i x_i \quad , \quad B = U(b) = \sum_{i=1}^{n} w_i y_i$$

The usual ordering of the numbers A and B induces an ordering of actions a and b (generally an ordering on the set of the actions).

On the other hand, in the preference-based methods first a and b are compared from the perspective of each c_i. This is usually done by calculating a preference index p_i

$$p_i = p(x_i, y_i)$$

with the help of some function p. The next step is the aggregation of the p_i-s:

$$P(a,b) = \sum_{i=1}^{n} w_i p(x_i, y_i)$$

To be able to compare the two approaches let us use the same function p and the aggregated utility values A and B to obtain another preference index $p(A, B)$ of actions a and b.

Theorem 1.

$$P(a,b) = p(A,B) \quad \text{iff} \quad p(x,y) = \frac{1}{2}(y - x + 1) \tag{1}$$

where $p(x, y)$is the linear form of the universal preference function.

Proof.

$$p(A,B) = \frac{1}{2}(B - A + 1) = \frac{1}{2}\left(\sum_{i=1}^{n} w_i y_i - \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} w_i\right) =$$
$$= \sum_{i=1}^{n} w_i \frac{y_i - x_i + 1}{2} = \sum_{i=1}^{n} w_i p(x_i, y_i) = P(a,b).\square$$

Another way of comparing the two approaches is by calculating for each action the value

$$V(a) = \frac{1}{n} \sum_{b} P(b,a)$$

where n is the number of actions, and p is the preference function given in the above theorem.

Theorem 2.

$$V(a) > V(b) \quad \text{iff} \quad U(a) > U(b). \tag{2}$$

Proof. Let $a_1, a_2, \ldots, a_m$ alternatives, $c_1, c_2, \ldots, c_n$ criteria and e_{ki} are utilities of a_k from the perspective of c_i, respectively. For $V(a_k)$ we get:

$$\begin{aligned} V(a_k) &= \frac{1}{n}\sum_{x=1}^{n} p(a_k, a_x) = \frac{1}{n}\sum_{x=1}^{n}\sum_{i=1}^{m} w_i \frac{e_{ki} - e_{xi} + 1}{2} = \\ &= \frac{1}{2n}\sum_{x=1}^{n}\sum_{i=1}^{m} w_i (e_{ki} - e_{xi} + 1) \end{aligned} \tag{3}$$

and for

$$\begin{aligned} V(a_k) - V(a_l) &= \frac{1}{2n}\sum_{x=1}^{n}\left(\sum_{i=1}^{m} w_i \left((e_{ki} - e_{xi} + 1) - (e_{li} - e_{xi} + 1)\right)\right) = \\ &= \frac{1}{2n}\sum_{x=1}^{n}\left(\sum_{i=1}^{m} w_i e_{ki} - \sum_{i=1}^{m} w_i e_{li}\right) = \frac{1}{2}\left(U(a_k) - U(a_l)\right). \square \end{aligned} \tag{4}$$

The function in (1) has some useful properties:

$$p(1,0) = 0, \quad p(0,1) = 1, \quad p(x,x) = \frac{1}{2},$$
$$p(x,y) + p(y,x) = 1.$$

Let the negation function $n(x) = 1 - x$. Then

$$p(x,y) + p(n(x), n(y)) = 1.$$

The next result gives the generalization of the theorem, using the aggregation operator

$$A = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right)$$

axiomatized by Dyckhoff [1] and Fodor [2], where f is the generator function of the aggregation, $f : [0,1] \to [0,1]$ continuous and strictly monotone increasing. The generalized negation function is:

$$n(x) = f^{-1}(1 - f(x)) \tag{5}$$

Theorem 3.

$$\textit{If} \quad p^*(x,y) = f^{-1}\left(\frac{1}{2}(f(x) - f(y) + 1)\right) \quad \textit{then} \quad p^*(a,b) = p(A,B) \tag{6}$$

where $p^*(x,y)$ *is the universal preference function.*

In other words, the preference and the aggregation are interchangeable.

$$p(U(a), U(b)) = U(p(a,b)).$$

Proof.

$$\begin{aligned}
p^*(a,b) &= f^{-1}\left(\frac{1}{2}(f(b)-f(a)+1)\right) = \\
&= f^{-1}\left(\frac{1}{2}\left(f\left(f^{-1}\left(\sum_{i=1}^{m} w_i f(y_i)\right)\right) - \right.\right. \\
&\quad \left.\left. - f\left(f^{-1}\left(\sum_{i=1}^{m} w_i f(x_i)\right)\right) + 1\right)\right) = \\
&= f^{-1}\left(\frac{1}{2}\left(\sum_{i=1}^{m} w_i f(y_i) - \sum_{i=1}^{m} w_i f(x_i) + \sum_{i=1}^{m} w_i\right)\right) = \\
&= f^{-1}\left(\frac{1}{2}\sum_{i=1}^{m} w_i (f(y_i) - f(x_i) + 1)\right). \square
\end{aligned}$$

The properties (2), (3), (4) hold for p^* as well with the modification

$$p^*(x,x) = \nu,$$

where ν is the neutral value of $n(x)$, i.e. $n(\nu) = \nu$.

2 Outranking Approach Based on the Universal Preference Function

In "The Outranking Approach and the Foundation of ELECTRE Methods" [3], Bernard Roy presents the fundamental aspects of this approach: the basic principles in the construction of the outranking relation, the concepts of the concordance and discordance. In this paper we are concentrating only to the concordance relation. For validating a comprehensive outranking relation S, it is necessary to take into account the fact that the role which has to be devoted to each criterion in the aggregation procedure is not necessarily the same. In other words, we need to characterize what is usually referred to as "the greater or less importance" give to each criterion. In methods, the importance of the j-th criterion is taken into account. By definition, the concordance index $c(a,b)$ characterizes the strength of the positive arguments able to validate the assertion aSb. By definition in ELECTRE I method

$$c_1(a,b) = \sum_{j \in aSb} w_j$$

and in ELECTRE II method

$$c_2(a,b) = \sum_{j \in aSb} w_j \frac{p_j + q_j(a) - g_j(b)}{p_j - q_j}$$

where

$$\sum_{i=1}^{n} w_i = 1$$

The outranking method of the PROMETHEE method is quite similar to the concordance index in the ELECTRE method. In order to estimate the preferences the decision maker is offered a choice, for each criterion between six forms of curves presented in [4]. In function of the way his preference increase with difference b-a, the decision maker sets, for each criterion, the form of the function and associated parameter(s). The parameters to be estimated have simple interpretation since they are indifference and preference thresholds.

The outranking methods are not preference-based methods in the above sense (i.e. universal preference function), because in the ELECTRE methods [3]

$$p(x,x) = 1$$

and in the PROMETHEE methods [4]

$$p(x,x) = 0.$$

According to (5) and (6) PROMETHEE and ELECTRE seem to be dual approaches. Furthermore, $p(x,y)$ and $p(y,x)$ are not related. The main reason is that $p(x,y)$ differs from the form (1) and the function $p(x,y)$ varies from criterion to criterion, i.e.

$$p(a,b) = \sum_{i=1}^{n} w_i p_i(x_i y_i).$$

We define the general form of the outranking and the utility approach:

$$p(a,b) = \sum_{i=1}^{n} w_i \tau_i(p_i(x_i y_i)). \tag{7}$$

where $\tau_i : [0,1] \to [0,1]$ are unary functions.

The next result presents the two outranking methods in a unified way.

Theorem 4. *For the preference function p^{EL},p^{PR} of the ELECTRE and PROMETHEE methods there exist unary functions τ_i^{EL} and τ_i^{PR} such that*

$$p^{EL}(a,b) = \sum_{i=1}^{n} w_i \tau_i^{EL}(p(x_i, y_i))$$
$$p^{PR}(a,b) = \sum_{i=1}^{n} w_i \tau_i^{PR}(p(x_i, y_i))$$

and moreover in the sense of theorem 2 we can get the utility approach from the general form (7).

Proof. Let $\tau^{EL}(x)$ and $\tau^{PR}(x)$ be the following functions:

$$\tau_i^{EL}(x) = \begin{cases} 0 & x \le p_i \\ \dfrac{x - p_i}{q_i - p_i} & p_i < x < q_i \\ 1 & x \ge q_i \end{cases}$$

for some $0 \le p_i \le q_i \le \frac{1}{2}$, and

$$\tau_i^{PR}(x) = \begin{cases} 0 & x \le p_i \\ \dfrac{x - p_i}{q_i - p_i} = \varphi(x) & p_i < x < q_i \\ 1 & x \ge q_i \end{cases}$$

for some $\frac{1}{2} \le p_i \le q_i \le 1$.

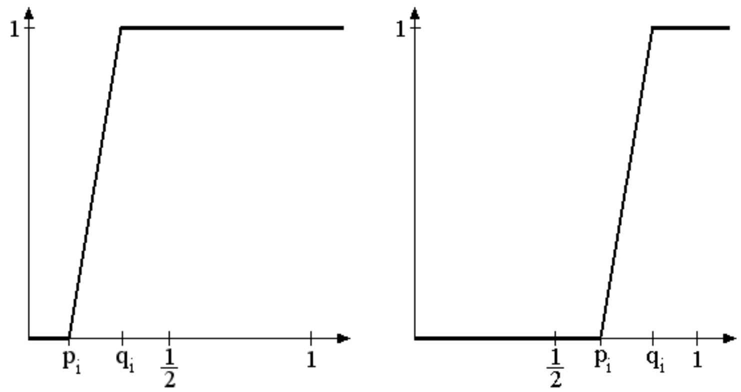

Fig. 1. τ^{EL} and τ^{PR} *functions*

The function $\varphi(x)$ might have different forms, see PROMETHEE [4].
The utility methods can be obtained by

$$\tau_i(x) = x.$$

We remark that if the τ function commute with the negation function i.e.

$$\tau_i(n(x)) = n(\tau(x))$$

then the properties (2), (3), (4) remain valid for p^*.

These constructions can be used to build a more general decision method choosing less restrictive functions for $\tau(x)$. It is incorporated into the Dec.Art method.

References

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