# Membership function as an evaluation

J. Dombi
Reseach Group of Theory of Automata

**Abstract**

After giving an overview of the different kinds of mathematical forms of the membership functions, we extract the different demands and determine the rational class of the membership functions. From the generalisation of our result, we can determine it giving four paramiters: The interval $[a, b]$, the sharpness $\lambda$, and the decision level $\nu$. We show the correspondence of this membership function to the Zimmermann-Zysno model, and using their empirical data, we also test our mathematical model. We also show the connections between the operators of the evaluation and membership functions, which gives the generalisation of this concept in a more general form. As a special case, we can get a linear form.

## 1 Introduction

The subject of fuzzy sets as an approach to the mathematical representation of everyday language was introduced by Zadeh [33] in 1965. In Zadeh's article, two important items can be found, i.e. the introduction of:

**(a)** the membership function and

**(b)** the many valued operators, working in this function.

In the development of fuzzy sets, a lot of articles were written dealing with different generalizations of the operators proposed by Zadeh, and only a few researchers have concentrated on the topic of membership functions. Now, most of the articles in journals dealing with fuzzy sets appear without using the membership function. It is always supposed that somehow we have many-valued inputs. All problems arising in the theory of fuzzy sets are due to the lack of our knowledge of the interpretations of "fuzzy". It is dangerous to neglect clarifying where the membership came from, because it is one of the most important aspects in the application. Here, we have an algorithm with a given input and we get the output as a result. How can we verify that the procedure is working well? It is not enough that the result is consistent with the reality, because we could reach it by manipulating the inputs, the membership functions. So for the question, "is the algorithm working?", this kind of proof gives no answer. By changing the existing input, the result also changes and this creates some contradictions. Input and algorithms have to be clearly connected.

Working without membership functions can be compared with dealing with probability theory where we have a calculus (or algebra) without probability density functions, and anybody using this theorem can arbitrarily choose the density functions. In this situation the statistical proofs are meaningless.

In section 2, we give a short overview of the different approaches of the membership function. In section 3, we describe the problem and summarise the demands on the basis of some critical ramarks. In section 4, we give the rational form of the membership functions. In section 5, we show how well this rational form fits in with empirical investigatmions. In section 6, we show the connections between the membership function and the operators.

Our main objectives were to find membership functions:

1. on a theoretical basis,
2. easy to calculate and fit to the problem,
3. described by only a few parameters
4. with parameters that are meaningful,
5. with a linearised form for the applications, and
6. with membership and operators closely connected.

We are not dealing with:

1. membership function systems,
2. linguistic hedges,
3. scaling problems,
4. measure problems,
5. context dependence,
6. uncertainty.

The answers to these kinds of questions can be found in another article on hedges, in the evaluations concept.

## 2 Different membership functions

The articles on membership functions can be classified in the following way:

**(I)** Heuristically based membership functions.

1. A function proposed by Zadeh [33][35][36][34] and later used by other authors; see Schwartz [27]:

$$\mu_{Young}(x) = \begin{cases} 1 & \text{if } x \leq 25, \\ \frac{1}{1+\left(\frac{x-25}{5}\right)^2} & \text{if } \ x > 25 \end{cases}$$

$$\mu_{Old}(x) = \begin{cases} 0 & \text{if } x \leq 50, \\ \frac{1}{1+\left(\frac{x-50}{5}\right)^{-2}} & \text{if } \ x > 50 \end{cases}$$

2. A function proposed by Krusinka and Liebhart [18]:

$$\mu(x) = \frac{1}{2} + \frac{1}{\pi}\text{arc } tg\frac{x-a}{b}, \ x \in [-\infty, \infty].$$

3. Functions proposed by Dimitru and Luban [5]

$$\mu(x) = \frac{1}{a^2}x^2 + 1, \quad x \in [0, a]$$

$$\mu(x) = \frac{1}{a^2}x^2 - \frac{2}{a}x + 1, \quad x \in [0, a]$$

4. A function proposed by Svarowski [30]:

$$\mu(x) = \frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right), \quad x \in [a, b]$$

**(II)** Membership functions based on reliability concerns with respect to the particular problem.

1. One of the most useful is the linear function introduced by Zimmermann [37], later used by several authors, e.g. Sakawa [24],

$$\mu(x) = 1 - \frac{x}{a}, \ x \in [0, a]$$

2. Other linear models are used by Heshmaty and Kandel [14] and Tanaka et. al. [10] It is a symmetrical function:

$$\mu(x) = \begin{cases} 1 - \frac{|\alpha - x|}{a} & \text{if } \alpha - a \leq x \leq \alpha + a, \\ 0 & \text{otherwise} \end{cases}$$

This model can be generalised for vectors.

3. When the membership function is not linear, then it can be linearised. A good example of this is the work of Hannan [11] and Sakawa and Yano[25]:

$$\mu(x) = \alpha\left(1 - e^{(b-x)(b-a)}\right), \quad x \in [a, b].$$

4. A function which is piecewise linear has been used by many authors, e.g. Bortolan and Degani [1], Chen [2].

$$\mu(x) = \begin{cases} 0 & \text{if } x \le a, \\ w_1 \frac{x-a}{b-a} & \text{if } a \le x \le b, \\ 1 & \text{if } b \le x \le c, \\ w_2 \frac{d-x}{d-c} & \text{if } c \le x \le d, \\ 0 & \text{if } d \le x. \end{cases}$$

5. Dimitru and Luban [4] used

$$\mu(x) = \frac{1}{1 + x/\alpha},$$

which fits the problem well.

**(III)** Membership functions based on more theoretical demand.

1. For decision making, Schwab [26] axiomatically justified membership functions on the basis of spline functions. The author used 11 axioms. The membership function has many parameters.
2. Civanlar ans Trussel [3] based the membership function on probability density functions $p(x)$

$$\mu(x) = \begin{cases} \lambda p(x) & \text{if } \lambda p(x) \le 1, \\ 1 & \text{if } \lambda p(x) > 1, \end{cases}$$

$\lambda$ can be solved from equations.

3. Svarovski [30] divides the membership functions into two parts and on the basis of some demands verifies the parameters:

$$\mu(x) = \begin{cases} 0 & \text{if } a \le a, \\ \mu_1(x) & \text{if } a \le x \le b, \\ \mu_1(x) & \text{if } b \le x \le c, \\ 1 & \text{if } c \le x \le d, \end{cases}$$

where $\mu_1(x) = K(x-a)^2$ and $\mu_2(x) = K_2x^2 + K_1x + K_0, K, K_0, K_1, K_2$ are parameters. The other approach is when we use $\mu_2(x) = K_3x^3 + K_2x^2 + K_1x + K_0$.

**(IV)** Membership functions and control.

In the area of control, it is necessary to use membership functions. Two different directions exist here:

1. either one defines the functions and identifies the system parameters,
2. or one works with a given system and idenrifies the membership function under the control process.

Sugeno et al. [28][29] belong to the former and Kiszka et al. [16][15] belong to the latter. It is important to note that there are works without parameter identifications e.g. Karwowski and Mital [17].

**(V)** Membership function as a model for human concepts.

The main aim here was to introduce fuzzy sets, e.g. Zadeh [36], to build a model of human concepts. It can be formally presented by fuzzy sets, e.g. Gougen [9]. Only a few articles deal with empirical research, namely:

1. Hersh et al. [12][13] In this study, the context effects upon the interpretation of a set of natural language terms (short, very short, sort of short, long, very long, sort of long). The membership function is:

$$\mu(x) = \frac{1}{2} + d\left(\frac{r}{10}\right) \quad \text{where} \quad d(x) = \begin{cases} 1 & \text{for yes responses,} \\ -1 & \text{for no responses} \end{cases}$$

   and $r =$ confidence value.

2. Norwich and Turksen [20] and Turksen [31] are critical of the fuzzy set theorusts becausee disagreement exist amongst them, especially regarding the measurement of fuzziness and the properties of the resulting membership functions. For example, Thole et al. [32] have stated that membership is on an absolute scale, while Saaty [23] has espoused a rational scale. Sticha et al. [22] claim an interval scale for membership but Gougen [9] has stated thet no stronger scale than ordinal or maybe interval scale measures the membership function well, and the general model is:

$$\mu_{m_A}(x) = f_m\left(\mu_A\left(g_m(x)\right)\right)$$

   where $f_m$ and $g_m$ describe linguistic hedges. Norwich presented their suggestion within measurement theory.

3. Zysno [39] and Zimmermann and Zysno [38] presented a model for determining the measurement function. Membership is defined as a function of the distance $d(x)$ between a given object $x$ and $x$ standard (ideal):

$$\mu(x) = \frac{1}{1 + d(x)}.$$

   Hence $d(x) = 0 \Longrightarrow \mu(x) = 1$ and $d(x) = \infty \Longrightarrow \mu(x) = 0$.
   For $d(x)$ was used:

$$d(x) = \frac{1}{e^{a(x-b)}}.$$

   It has the additional advantage that it can easily be linearised by transformation, and easy to check its validity. The complete model of membership is

$$\mu(x) = \left[\frac{1}{d}\left(\frac{1}{1+e^{-a(x-b)}} - c\right) + \frac{1}{2}\right]$$

where the square brackets indicates that values always are in the interval $[0,1]$. The formal description is the following:

$$[x] = \begin{cases} 1 & \text{if} \quad 1 < x \\ x & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{if} \quad x < 0 \end{cases}$$

After finishing this summary, we have to mention that we are concentrating on articles which have appeared in the last 3 years or were first printed in Fuzzy Sets and Systems and we do not intend to consider all articles in which the membership function appears.

In this sense, our list may be extended, but it was not our aim to make a comprehansive literature survey on membership functions.

# 3 Properties for constructing the membership function

It is not easy to find common features among these different approaches, although we find that every article:

1. all membership functions are continuous,
2. all membership functions map an interval $[a,b]$ to $[0,1]$, $\mu[a,b] \longrightarrow [0,1]$.
3. the membership functions are (a) either monotonically increasing, or (b) monotonically decreasing, or (c) could be divided into a monotonically increasing or decreasing part.
4. the monotonous membership functions on the whole interval are (a) either convex functions or (b) concave functions, or (c) there exists a point $c$ in the interval $[a,b]$ such that $[a,c]$ is convex and $[c,b]$ is concave (called S-shaped functions). In most of the articles in which the membership function does not appear in explicit form the author supposes that it has an S-shaped form, e.g. Lakoff [19], Norris et al.[21], Gougen [9], and Zadeh [34].
5. monotonically increasing functions have the property $u(a) = 0$, $u(b) = 1$, while monotonically decreasing functions have the property $u(a) = 1$, $u(b) = 0$.
6. very important is the linear form or linearisation of the membership function.

7. the Zimmermann-Zysno and the Dimitru-Luban concept are two of the most remarkable. It is interesting that Zadeh's function also has the same form. Hersch also used a membership function in the sense of distance.

   The ppoposed membership functions are not good enough because:

   (a) they cannot be generalised to other cases (Zadeh, Karwowski)
   (b) some of them are too general (Norwich)
   (c) the parameters have no meaning (Svarovski)
   (d) the pareameters cannot easily be calculated (Zimmermann-Zysno)

   From the Zimmermann-Zysno model we can express the interval $[A, B]$ where the function is between 0 and 1:

$$A = \ln\left(e^{-b}\left(\frac{2-2c+d}{2c-d}\right)^{1/a}\right),$$

$$B = \ln\left(c^{-b}\left(\frac{2-2c-d}{2c-d}\right)^{1/a}\right).$$

   If interval $[A, B]$ is known, then from the above form, it is easy to determine the parameters $a, b, c, d$. This is one of the disadvantages of this kind of membership function. The model suggested by Zimmermann and Zysno principally contains variables of a psychological nature. In our work, we chaged the fundamental assumption of this kind.

Now we can make some reductions in the ways of determination of the membership functions.

To build a membership function which has monotonically increasing and decreasing parts, we can use the $c(x, y)$ conjunctive operator. Because all kinds of conjunctive operators have the property $c(x, 1) = x$ and letting

$$\mu_A^*(x) = \begin{cases} 0 & \text{if } x \le a, \\ \mu_A^*(x) & \text{if } a < x \le b, \\ 1 & \text{if } b < x \end{cases}$$

and

$$\mu_B^*(x) = \begin{cases} 0 & \text{if } x \le c, \\ \mu_B^*(x) & \text{if } c < x \le d, \\ 1 & \text{if } d < x \end{cases}$$

then

$$\mu_{AB}(x) = c\left(\mu_A^*(x), \mu_B^*(x)\right).$$

every function $f(x)$ ordered in the interval $[a, b]$ can be transformed to another interval $[A, B]$ with the help of the following mapping:

$$y=\frac{x-a}{b-a}\left(B-A\right)+A,\qquad x\in\left[a,b\right].$$

From monotonically increasing membership functions, we get a monotoically decreasing function using the linear transformation

$$y=\frac{b-x}{b-a}\left(B-A\right)+A,\qquad x\in\left[a,b\right].$$

After this consideration, our task is to determine a membership function which is continuously monotonically increasing $\mu:\left[0,1\right]\longrightarrow\left[0,1\right]$. Then from $\mu$, using the linear transformation we get:

1.
$$\mu_{[a,b]}\left(x\right)=\begin{cases}\mu_{[0,1]}\left(\frac{x-a}{b-a}\right) & \text{if } \mu \text{ is monotonically increasing}\\ \mu_{[0,1]}\left(\frac{b-x}{b-a}\right) & \text{if } \mu \text{ is monotonically decreasing}\end{cases}$$

# 4 Mathematical form of the membership function

In view of the ideas put forward in Section 3, the membership function should meet the following demands:

**C1.** $\mu\left(x\right)$ is a continuously increasing function $\mu\left(x\right):\left[0,1\right]\longrightarrow\left[0,1\right]$.

**C2.** $\mu\left(0\right)=0$, $\mu\left(1\right)=1$ (boundary condition)

**C3.** $\mu'\left(0\right)=0$, $\mu'\left(1\right)=0$ (S-shaped character)

**C4.** $\mu\left(x\right)$ is a rational function of polynomials

$$\mu\left(x\right)=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{A_0x^m+A_1x^{m-1}+\cdots+a_m}\qquad\left(m\neq 0\right).$$

**C5.** Find such kind of $\mu\left(x\right)$ where $n+m$ is minimum.

**Theorem 1** *There are no membership functions fulfilling the properties C1-C5 if $n+m\leq 3$.*

**Proof.** (1) Let us suppose that there are parameters $a,b,A,B$ so that

$$\mu\left(x\right)=\frac{ax+b}{Ax+B}$$

fulfils the demands C1-C5. Because $\mu\left(0\right)=0$, we have $b=0$ and because $\mu\left(1\right)=1$, we have $a=A+B$. Hence $\mu\left(x\right)=\left(A+B\right)x/\left(Ax+B\right)$. Calculating $\mu'\left(x\right)$, we obtain

$$\mu'\left(x\right)=\frac{\left(A+B\right)\left(Ax+B\right)-A\left(A+B\right)x}{\left(Ax+B\right)^2}$$

and from $\mu'(0) = 0$ it follows that $(A+B)B = 0$. Now if $B = 0$, then $\mu(x) \equiv 1$, and if $A + B = 0$, then $\mu(x) \equiv 0$, and this is in contradiction to C2. ■

**Proof.** (2) Let us suppose that there are parameters $a, b, A, B$ so that

$$\mu(x) = \frac{ax+b}{Ax^2+Bx+C}.$$

From $\mu(0) = 0$ it follows that $b = 0$. Calculating $\mu'(x)$, we obtain

$$\mu'(x) = \frac{\left(Ax^2+Bx+C\right) - ax(2Ax+B)}{\left(Ax^2+Bx+C\right)^2}$$

Because $\mu'(0) = 0$ we have $c = 0$. Because $a \neq 0$ then $C = 0$ and $\mu(x) = a/(Ax+B)$. Part (1) above shows that this is a contradiction. ■

**Proof.** (3) Let us suppose that there are parameters $a, b, A, B$ so that

$$\mu(x) = \frac{ax^2+bx+c}{Ax+B}$$

Because $\mu(0) = 0$, $c = 0$. Calculating $\mu'(x)$ we obtain

$$\mu'(x) = \frac{(Ax+B)(2ax+b) - A\left(ax^2+bx\right)}{(Ax+B)^2}$$

Because $\mu'(0) = 0$ we have $Bb = 0$.

Now if $B = 0$ then

$$\mu'(x) = \frac{a}{A}x + \frac{b}{A}$$

and this is a contradiction.

If $b = 0$ then

$$\mu(x) = \frac{ax^2}{Ax+B}.$$

Because $\mu(1) = 1$ we have $a = A + B$ and $\mu'(1) = 0$ so $2(A+B) = A$. From this, $A = -B$ and $a = 0$, which is still a contradiction. ■

**Theorem 2** *The minimum of $n+m$ is 4 and the membership function is*

$$\mu(x) = \frac{(1-\nu)x^2}{(1-\nu)x^2 + \nu(1-x)^2}$$

*where $\nu$ is the intension value of $y = \mu(x)$ and $y = x$ so $\nu$ is the characteristic value of the shape.*

**Proof.** Let

$$\mu(x) = \frac{ax^2 + bx + c}{Ax^2 + Bx + C}.$$

Because $\mu(0) = 0$, $c = 0$. Calculating $\mu'(x)$ we get

$$\mu'(x) = \frac{\left(Ax^2 + Bx + C\right)(2ax + b) - (2Ax + B)\left(ax^2 + bx\right)}{\left(Ax^2 + Bx + C\right)^2}$$

Because $\mu'(0) = 0$ we have $Cb = 0$. Now $C \neq 0$ because of Theorem 1, so $b = 0$ and

$$\mu(x) = \frac{ax^2}{Ax^2 + Bx + C}.$$

Because $\mu(1) = 1$ and $\mu'(1) = 0$, we get $a = A + B + C$ and $b = -2C$. So we obtain

$$\mu(x) = \frac{(A - C)\,x^2}{Ax^2 - 2Cx + C},$$

where $a, A, C$ are free parameters but $A \neq C$. This form is equivalent to

$$\mu(x) = \frac{x^2}{x^2 + \alpha\,(1 - x)^2}$$

where $\alpha = C/\,(A - C)\,.$

We must show that $\alpha = \nu/\,(1 - \nu)$ where $\nu = \mu(\nu)\,.$ From the equation

$$\nu = \frac{\nu^2}{\nu^2 + \alpha\,(1 - \nu)^2}$$

we can derive the required results. ■

The function $\mu(x)$ depends on the parameter $\nu$. The membership function can be presented in this form:

$$\mu(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\left(\frac{1-x}{x}\right)^2} \tag{1}$$

This is the same form as was proposed by Zimmermann and Zysno [38]. Membership is defined here as a function of the distance $d(x)$ between a given object and a standard (ideal). Hence, $d(x) = 0$ implies $\mu(x) = 1$ and $d(x) = \infty$ implies $\mu(x) = 0$. or $\mu(x) = 1/\,(1 + d(x))\,.$

Our distance function is

$$\frac{\nu}{1 - \nu}\left(\frac{1 - x}{x}\right)^2.$$

Let us generalise this $d(x)$ function in the following way:

$$d\left(x\right)=\left(\frac{\nu}{1-\nu}\right)^{\lambda-1}\left(\frac{1-x}{x}\right)^{\lambda}. \tag{2}$$

**Theorem 3** *The membership function using the distance (2) is*

$$\mu\left(x\right)=\frac{\left(1-\nu\right)^{\lambda-1}x^{\lambda}}{\left(1-\nu\right)^{\lambda-1}x^{\lambda}+\nu^{\lambda-1}\left(1-x\right)^{\lambda}},\quad \lambda>1, \tag{3}$$

*where $\mu\left(\nu\right)=\nu$ and $\mu'\left(\nu\right)=\lambda$, and it fulfils the demands C1-C4, but not C5. Using linear transformation, the membership function on interval (a,b) is*

$$\mu\left(x\right)=\frac{\left(1-\nu\right)^{\lambda-1}\left(x-a\right)^{\lambda}}{\left(1-\nu\right)^{\lambda-1}\left(x-a\right)^{\lambda}+\nu^{\lambda-1}\left(b-x\right)^{\lambda}} \tag{4}$$

*and the distance function $d\left(x\right)$ in this case is*

$$d\left(x\right)=\left(\frac{\nu}{1-\nu}\right)^{\lambda-1}\left(\frac{b-x}{x-a}\right)^{\lambda}.$$

*The transformed value $\nu$ is*

$$x_{\nu}=\left(b-a\right)\nu+a$$

*and*

$$\lambda=f'\left(x_{\nu}\right)\left(b-a\right)$$

*which is an indicator of increasing membership. This is the sharpness.*

*The monotonously decreasing functions obtained using the linear transformation are*

$$\mu\left(x\right)=\frac{\left(1-\nu\right)^{\lambda-1}\left(b-x\right)^{\lambda}}{\left(1-\nu\right)^{\lambda-1}\left(b-x\right)^{\lambda}+\nu^{\lambda-1}\left(x-a\right)^{\lambda}} \tag{5}$$

*with $\mu\left(x_{\nu}\right)=\nu$ and $x_{\nu}=\left(a-b\right)\nu+b$.*

**Proof.** C1-C4 are obviously fulfilled. Calculating $\mu'\left(x\right)$, we obtain

$$\mu'\left(x\right)=\lambda\left(b-a\right)\frac{\left(1-\nu\right)^{\lambda-1}\left(x-a\right)^{\lambda-1}\nu^{\lambda-1}\left(b-x\right)^{\lambda-1}}{\left(\left(1-\nu\right)^{\lambda-1}\left(x-a\right)^{\lambda-1}+\nu^{\lambda-1}\left(b-x\right)^{\lambda}\right)^{2}}$$

From this, $\mu'\left(a\right)=0$ and $\mu'\left(b\right)=0$ are also valid and $\mu'\left(x_{\nu}\right)=\lambda/\left(b-a\right)$. However, $x_{\nu}$ is not the inflection point, or $\mu''\left(x_{\nu}\right)\neq0$, if $\nu\neq0.5$;

$$\mu''\left(x_{\nu}\right)=\frac{\lambda\left(\lambda-1\right)}{\left(b-a\right)^{2}}\frac{1-2\nu}{\nu\left(1-\nu\right)}.$$

■

Now the determination of any membership function is possible with four parameters: first, we should give the interval $(a,b)$. The other two parameters also have meaning: $\lambda$ is the sharpness of the membership function and $\nu$ could be interpreted as an expectation level. To make these last remarks clear, an interesting form of the rational class of negations [6] is

$$n(x) = \frac{\nu^2 (1-x)}{\nu^2 (1-x) + (1-\nu)^2 x}$$

where $n(\nu) = \nu$ and $\nu$ is the turning point. If an $x$ is small, then $\nu$ will be greater utilizing the negation, and vice versa. The fixed point of the negerion is $\nu$. It does not change. In this work, we considered or interpreted the membership as an evaluation though the negation is a transformation from bad values to good values and vice versa. Such a point should also exist on the membership functions and we defuned it as the $\nu$ value. Figures 1 and 2 show the influence on the membership function of changing the parameters $\nu$ and $\lambda$. On Figure 3, we can see the first and second derivative also.

## 5 Empirical evidence

We used Zysno's data published in [39]. In this project, 64 subjects (16 for each set) from 21 to 25 years of age rated 52 different statements of age concerning one of the four fuzzy sets "very young man", "young man", "old man" and "very old man".

From the point of view of linearisation, membership function (4) has the additional advantage that it can easily be linearised by the following transformation:

$$\ln\left(\frac{1-\mu}{\mu}\right) = \lambda \ln\left(\frac{b-x}{x-a}\right) + (\lambda - 1)\ln\left(\frac{\nu}{1-\nu}\right)$$

On the basis our membership values are transformed into $y$ values and the $x$ values are transformed too:

$$y = \ln\left(\frac{1-\mu}{\mu}\right), \qquad x = \ln\left(\frac{b-x}{x-a}\right).$$

We used the following notations:

$$c = \ln\left(\frac{\nu}{1-\nu}\right),$$

so the membership function is

$$y = \lambda x + (\lambda - 1)c = \lambda x + d$$

where $d = (\lambda - 1)c$.

We have supposed that the parameters $a, b$ can be easilz identified and from the linearised form, via the sami method as was used by Zimmermann and Zysno, we estimated the $d$ and $c$ value. From $c$ we get

$$\nu = \frac{e^c}{1 + e^c}$$

When we want to compare with a linear form and $\nu, \lambda$ are known, then we should use the transformation

$$y = \frac{(1-\nu)^{(1/\lambda)-1} \mu^{1/\lambda}}{(1-\nu)^{(1/\lambda)-1} \mu^{1/\lambda} + \nu^{(1/\lambda)-1} (1-\mu)^{(1/\lambda)-1}}$$

where $y = \mu(x)$. So the transformed membership function could be written in the form

$$y = \frac{x-a}{b-a}.$$

Using this form, $(a, b)$ can be determined if $\nu$ and $\lambda$ are known.

The evaluation of the data shows a good fit to the model. Figures 4-8 show membership functions given by different respondents. The estimation error has the same degree as in [38].

# 6 Connection of the membership function with operators

As has already been shown, the data and operators working on it cannot be independent. The question is the meaningfulness. In this section, we show the connection between the rational class of membership functions and the rational form of the aggregation operators given by Dombi [6].

**Definition 4** *An aggregation operator is a continuous and strictly monotonically increasing function* $a : [0,1] \longrightarrow [0,1]$ *satisfying:*

**(1)** $a(0,0) = 0$

**(2)** $a(1,1) = 1$

**(3)** $a(x, a(y,z)) = a(a(x,y), z)$ *(associative)*

**(4)** $a(x,y) = n(a(n(x), n(y)))$ *(self-distributive)*

**Theorem 5** *[6] The rational class of aggregation operators has the form*

$$a(x,y) = f^{-1}(f(x) + f(y))$$

*where*

$$f(x) = \ln\left(\frac{1-\nu}{\nu} \frac{x}{1-x}\right)$$

**Theorem 6** *The rational class of membership functions (3) has the following form:*

$$\mu_{[0,1]}(x) = f^{\ -1}(\lambda f(x)) \tag{6}$$

*where $f(x)$ is the generator function of the rational aggregation operator.*

**Proof.** Calculating

$$f^{\ -1}(x) = \frac{\nu e^x}{(1-\nu)+\nu e^x}$$

and using the above theorem, we get the result. ■

The new interpretations of the membership functions open up new vistas. One very useful class of continuous valued logical and set-theoritical operators, the t-norms and t-conorms, can be represented with the help of generator functions. A lot of articles deal with t-norms and t-conorms [7][8]. We distinguish two types: the strict and the non-strict operators. We can describe them as follows:

**Definition 7** *The brakets* [] *have the following meaning:*

$$[x] = \begin{cases} 1 & if\ \ x \geq 1 \\ x & if\ \ 0 < x < 1 \\ 0 & if\ \ x \leq 0 \end{cases}$$

**Theorem 8** *[3] A continuous strictly monotonically increasing operator on* $(0,1)\times(0,1)$ *with the boundary conditions*

$$\begin{aligned} c(x,1) &= x, \qquad d(x,1) = 1, \\ c(x,0) &= 0, \qquad d(x,0) = x, \end{aligned}$$

*which is Archimedean,*

$$\begin{aligned} c(x,x) &< x, \qquad x \in (0,1) \\ d(x,x) &> x, \qquad x \in (0,1) \end{aligned}$$

*and associative,*

$$\begin{aligned} c(x,c(y,z)) &= c(c(x,y),z) \\ d(x,d(y,z)) &= d(d(x,y),z) \end{aligned}$$

*can always be written in the following form:*

$$\begin{aligned} c(x,y) &= f^{\ -1}(f(x)+f(y)) \\ d(x,y) &= g^{-1}(g(x)+g(y)) \end{aligned}$$

*where $f : [0,1] \longrightarrow R$ $(g : [0,1] \longrightarrow R)$ are continuous strictly monotonically decreasing (increasing) functions and $f(x) \longrightarrow \infty$ if $x \longrightarrow 0$ and $g(x) \longrightarrow \infty$ if $x \longrightarrow 1$.*

**Proof.** See [7] ■

**Theorem 9** *[3] If the operators are not strictly increasing with the above conditions, then*

$$\begin{aligned} c(x,y) &= f^{-1}[f(x)+f(y)] \\ d(x,y) &= g^{-1}[g(x)+g(y)] \end{aligned}$$

*where $f : [0,1] \longrightarrow [0,1]$ is a strictly monotonically increasing function with $f(0) = 1$, $f(1) = 0$ and $g : [0,1] \longrightarrow [0,1]$ strictly monotonically increasing function with $g(0) = 0$, $g(1) = 1$.*

**Proof.** See [6] ■

**Definition 10** *The conjunctive (disjunctive) type of membership functions are in the strict and not strict case, respectively:*

$$\begin{aligned} \mu_c(x) &= f^{-1}\left(\lambda f\left(\frac{x-a}{b-a}\right)\right) \\ \mu_c(x) &= f^{-1}\left[\lambda f\left(\frac{x-a}{b-a}\right)\right] \end{aligned}$$

$$\begin{aligned} \mu_d &= g^{-1}\left(\lambda g\left(\frac{x-a}{b-a}\right)\right) \\ \mu_d(x) &= g^{-1}\left[\lambda g\left(\frac{x-a}{b-a}\right)\right] \end{aligned}$$

**Remark 11**

1. *If $\lambda = 1$, then the membership function is linear.*
2. *If $c(x,y) = xy$, then $\mu_{[0,1]} = x^{\lambda}$. See figure 9.*
3. *If $d(x,y) = x+y-xy$, then $\mu_{[0,1]}(\lambda) = 1-(1-x)^{\lambda}$. See figure 10.*
4. *If $c(x,y) = [x+y-1]$ then $\mu_{[0,1]}(x) = 1-[\lambda(1-x)]$. See figure 11.*
5. *If $d(x,y) = [x+y]$ then $\mu_{[0,1]}(x) = [\lambda x]$. See figure 12.*

## 7 Conclusions

In this paper we have given a new class of membership functions. It can be described with only four parameters. All four parameters are easy to determine.The first two determine the interval $[a, b]$ , $\lambda$ is the sharpness, and $\nu$ determines the inflection point of the following S-shaped functions (see figures 1,2 ): the monotonically increasing function

$$\mu(x) = \frac{(1-\nu)^{\lambda-1}(x-a)^{\lambda}}{(1-\nu)^{\lambda-1}(x-a)^{\lambda} + \nu^{\lambda-1}(b-x)^{\lambda}}, \quad x \in [a, b]$$

and the monotonically decreasing function

$$\mu(x) = \frac{(1-\nu)^{\lambda-1}(b-x)^{\lambda}}{(1-\nu)^{\lambda-1}(b-x)^{\lambda} + \nu^{\lambda-1}(x-a)^{\lambda}}, \quad x \in [a, b]$$

If $\lambda = 1$, then we get the linear case

$$\mu(x) = \frac{x-a}{b-a}$$

It has the same form as that proposed by Zimmermann and Zysno, Zadeh and others:

$$\mu(x) = \frac{1}{1+d(x)}.$$

We have shown the connection of these types of membership functions and the aggregation operator.

In the increasing case,

$$\mu_c(x) = f^{-1}\left[\lambda g\left(\frac{x-a}{b-a}\right)\right], \quad x \in [a, b]$$

and in the decreasing case

$$\mu_D(x) = f^{-1}\left[\lambda f\left(\frac{b-x}{b-a}\right)\right], \quad x \in [a, b].$$

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