European Journal of Operational Research 26 (1986) 35–41
North-Holland

A method for determining the weights of criteria: The centralized weights

Tamás SOLYMOSI
Computing Centre of Szeged Medical University, H-6720 Szeged, Pécsi u. 4/a, Hungary

József DOMBI
Research Group on Theory of Automata, H-6720 Szeged, Somogyi u. 7, Hungary

Abstract: In multicriteria analysis many methods use weights to represent the relative importance of criteria. The noncompensatory aggregation procedure used in outranking methods enables comparisons of subsets of criteria from this point of view. An interactive method is presented, which requests only ordinal comparisons from the decision maker. The relation 'more important than' is assumed to be a semiorder. Therefore, the indifference part of this relation is not necessarily transitive. The judgements can be formulated as linear inequalities, which constrain the set of feasible weights and threshold values. All vertices of this polyhedron are determined, the threshold and the unknown real weights are estimated by the centroid. Some characteristics, a possible refinement in the estimation and an illustrative example are also given.

Keywords: Decision theory, linear programming

1. Introduction

In multicriteria analysis it is essential to determine the set of criteria, which represent the points of view taken into account in a decision problem. The points of view are generally in conflict and have different importances for the decision maker. Speaking about importances of viewpoints has meaning only in comparing them to one another. The information gathered from the decision maker concerning the relative importances, their representation, use and effect on the final result depend fundamentally on the aggregation procedures used in the methods. Many of them require the 'proportion' of importances represented as numbers, since they use these numbers to modulate additive quantities as weights.

In the additive utility model, for example, when the utility function is normalized, the importances of criteria appear as coefficients in a convex combination of the marginal utility functions. In assessing an additive utility function some methods, working on the basis of this model (see, for example, [6,9,15]), determine the weights of criteria as estimated parameters for the model to be as consistent as possible with known subjective preference judgements among the alternatives. In this case compensatoriness among the criteria is obviously assumed and the weights are in close connection with tradeoffs.

Most of outranking methods (ELECTRE I, II, III [11,12,13], PROMETHEE I, II, III [2,3]) require some quantification of the importances of criteria as input data, but the papers do not treat the determination of weights. (We know only the TACTIC method [17], where the weights are obtained by using linear programming.) The importances represented by nonnegative numbers are used to characterize and compare subsets of criteria independently of the evaluation of alternatives on the criteria. The notion of relative importances can be clearly defined in this case. The noncompensatory aggregation procedure (see [1]) makes this intrinsic

Received July 1985

0377-2217/86/$3.50

characterization appropriate (see [16]). In Section 3 we outline an algorithm, in which the weights are estimated from ordinal judgements concerning the importances of subsets of criteria. It can be useful for determining the weights in applications of these methods.

Determination of weights requires the setting of weighting scales, i.e. obtaining information from the decision maker to express as numbers not only the order of criteria according to their importances, but also the intensity of this relation. Obviously, each subset of criteria must be comparable. If the relation 'more important than' on the set of all subsets of criteria is assumed to be monotone, then the problem of setting a weighting scale means gathering information to eliminate the incomparability of the relation 'subset of' and the real valued representation of this relation. In an additive representation, the weights of the single criteria uniquely determine the weight of any subset as the sum of weights of the criteria contained; the monotonicity can be guaranteed by nonnegativity. Due to the relevance of the 'proportion' of importances the assumption of standardized weights is not restrictive.

From the previous works devoted to determine weights we mention only the Churchman–Ackoff method [4], in which the initial values derived from direct estimation are tested systematically by comparing ordinally subsets of criteria and in which the current values are modified, when inconsistency arises. Despite the imperfections of the method (for details and for a revised technique from the point of view of computerization see [8]), the algorithm used in the ordinal comparisons of subsets of criteria can be effectively utilized as an asking algorithm in building up a connected relation on the set of all subsets of criteria, if the relation is assumed to be monotone.

In this paper we present a method to determine the feasible set of weights and threshold values for the relation 'more important than', assumed to be a semiorder. We suggest to choose a centre from the feasible set and a mean value for the threshold in lack of further information. The numerical representation of judgements and conditions of applicability of the method (assuming a noncompensatory aggregation procedure) are discussed in the next section. In Section 3 we outline an algorithm to determine all vertices of the feasible set, weights and threshold values simultaneously. Section 4 describes the questioning procedure and some characteristics. At the end of the paper a possible refinement and an illustrative example are given.

2. Conditions of applicability and the model

We denote by $C=\{C_1,\dots,C_{n-1}\}$ $(n\geqslant 3)$ the finite set of criteria, by 2^C the set of all subsets of C, by aPb $(a,\, b\in 2^C)$ if the subset a is more important than the subset b and by aIb if there is no considerable difference between the importances of a and b.

The first assumption of the method is that in the decision maker's mind the importances of criteria (implicitly) exist, and he/she is able to express them in ordinal comparisons, i.e. there exists a W: $2^C\to\mathbb{R}_+=\{r\colon r\geqslant 0\}$ function assumed to be

- additive: $W(a\cup b)=W(a)W(b)$ *for all* $a,\, b\in 2^C,\ a\cap b=\emptyset$;
- standardized: $W(C)=1$.

The second assumption is that P is a semiorder on 2^C, i.e. irreflexive, $[(xPy)\wedge(zPw)$ implies $(xPw)\vee(zPy)]$ and $[(xPy)\wedge(yPz)$ implies $(xPw)\vee(wPz)]$ holds for all $x,\ y,\ z,\ w\in 2^C$. The transitivity and asymmetry of P and the reflexivity and symmetry of I (defined by xIy iff not $(xPy)\wedge$ not (yPx) for all $x,\ y\in 2^C$) follow from the assumption, but I is not necessarily transitive. The real-valued representation of P is aPb iff $W(a)>W(b)+\delta$ for all $a,\ b\in 2^C$ for some $\delta\geqslant 0$. If $R=P\cup I$ then we have a reflexive and connected relation with the representation aRb iff $W(a)\geqslant W(b)-\delta$ for all $a,\ b\in 2^C$. The relation I is transitive, iff $\delta=0$. In this case P is a weak order, see, for example [5].

Let us denote by w_j the estimated value of $W(C_j)$ $(j=1,\dots,n-1)$ and by w_n the threshold value $(w_n=\delta)$. For these values must hold:

$$w_j\geqslant 0 \quad (j=1,\dots,n),$$

$$\sum_{j=1}^{n-1} w_j=1. \tag{1}$$

During the process the feasible set of weights is restricted step by step according to the judgements given by the decision maker and a proper threshold value is determined simultaneously.

In each step two disjoint subsets—denoted by $a,\ b$—of C are compared and either aPb or bRa

holds. If we have m judgements we get a system of inequalities, since the i-th judgement ($i = 1, \dots, m$) can be formulated as

$$\sum_{j=1}^{n} a_{ij} w_j \leqslant 0,$$

where

$$a_{in} = \begin{cases} 1 & \text{if } aPb, \\ -1 & \text{if } bRa, \end{cases}$$

and

$$a_{ij} = \begin{cases} -a_{in} & \text{if } C_j \in a, \\ 0 & \text{if } C_j \notin a \cup b \quad (j = 1, \dots, n-1), \\ a_{in} & \text{if } C_j \in b. \end{cases}$$

Using the algorithm described in the next section we determine the vertices of

$$L = \left\{ w \in \mathbb{R}^n \colon\ w \geqslant \mathbf{0},\ Aw \leqslant \mathbf{0},\ \sum_{j=1}^{n-1} w_j = 1 \right\} \tag{2}$$

where $A = [a_{ij}]$ is an (m, n)-matrix with only -1, 0, 1 entries. (Questions, which would yield an inequality $2w_1 \leqslant 3w_2 + w_3$ would be entirely unrealistic.) The polyhedron L is the set of feasible weights and threshold values consistent with every judgement and each point of L can be obtained as a convex combination of the vertices. We can present all vertices to the decision maker to choose one or some of them, which are satisfactory, or to exclude the unsatisfactory ones. Due to the interactivity we can stop if we reach satisfactory values before the asking procedure described in Section 4 is finished.

However, in lack of further information we determine the centroid of L. The vertices can be viewed as the result of an immoderate strategy in selecting a point from the feasible set, when only some of the judgements are emphasized, while the others are neglected, and the threshold value is to be increased/decreased as possible. It follows from the algorithm and the integer entries of A that the vertices are often too regular, e.g. distances between different weights are equal, weights are often equal, the threshold value is either the possible greatest or equal to zero, (see the illustrating example) which are baseless in general. On the other hand, an optimal solution of a linear optimization (e.g. maximizing the threshold value—the 'distance' from the faces of the polyhedron) can alter significantly, if the constraints change (e.g. we take into account a new judgement). For these reasons we select the centroid, which is always an inner point, less sensitive for the change of judgements and provide a mean 'distance' from both the vertices and the hyperplanes. This point is called the centralized weights.

3. Determination of the centralized weights

Our task is the determination of all vertices of L defined by (2). If L is nonempty, these points 'coincide'—except for multiplication by a positive number—with the extremal directions of

$$K = \{ w\colon\ w \geqslant \mathbf{0},\ Aw \leqslant \mathbf{0} \}.$$

It is sufficient to require (1) at the end of the algorithm. K is a convex, closed cone, so we utilize the method of total description known from linear programming. The number of extremal directions is finite, since K is the intersection of finite, closed half-spaces.

In the 0-th step let

$$K_0 = \{ w\colon\ w \geqslant \mathbf{0} \}$$

and let Q_0 be an (n, n)-matrix, the columns of which are the extremal directions of K_0, i.e. the n unit vectors. We take the judgements into account step by step, and determine the extremal directions of the set of feasible solutions. This interactivity enables us to follow a dynamic asking algorithm, thus, the number of judgements—denoted by m—is not known at the beginning.

In the i-th step ($i = 1, \dots, m$) we already know the cone

$$K_{i-1} = \{ w\colon\ w \geqslant \mathbf{0},\ A_{i-1} w \leqslant \mathbf{0} \} \tag{3}$$

and the (n, q_{i-1})-matrix $Q_{i-1} = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_q]$ ($q = q_{i-1}$) which—let us suppose—is not the zero-matrix and contains the extremal directions of $\lfloor K_i - 1 \rfloor$. If the i-th judgement can be formulated as $\boldsymbol{a}_i^{\mathrm{T}} w \leqslant 0$, then $A_{i-1}^{\mathrm{T}} = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_{i-1}]$ and A_0 is a 'matrix' which has no entry. Form the vector

$$\boldsymbol{b}_i^{\mathrm{T}} = \boldsymbol{a}_i^{\mathrm{T}} Q_{i-1}.$$

Let $\boldsymbol{e}_s$ denote the q_{i-1}-dimensional unit vectors ($s = 1, \dots, q_{i-1}$) and let

$$N_+ = \{ r\colon\ \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_r > 0 \} = \{ r_1, \dots, r_R \},$$

$$N_0 = \{ s\colon \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_s = 0 \} = \{ s_1, \dots, s_S \},$$
$$N_- = \{ t\colon \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_t < 0 \} = \{ t_1, \dots, t_T \}$$

be the classes of indices according to the sign of the corresponding entries of $\boldsymbol{b}_i$. We write $\boldsymbol{e}_s$, N_+, N_0, N_- instead of the more exact $\boldsymbol{e}_s^{(i)}$, $N_+^{(i)}$, $N_0^{(i)}$, $N_-^{(i)}$, but this will not give any conclusion. Let us construct

$$P_i = \left[\boldsymbol{e}_{s_1}, \dots, \boldsymbol{e}_{s_S}, \boldsymbol{e}_{t_1}, \dots \boldsymbol{e}_{t_T}, \boldsymbol{e}_{t_1 r_1}, \dots, \boldsymbol{e}_{t_T r_R} \right],$$

the (q_{i-1}, p_i)-matrix, where $p_i = S + T + RT$ and

$$\boldsymbol{e}_{tr} = \left(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_r \right) \boldsymbol{e}_t - \left(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_t \right) \boldsymbol{e}_r$$

for all $r \in N_+$, $t \in N_-$. If $p_i = 0$, i.e. $N_0 \cup N_- = \emptyset$, $\boldsymbol{b}_i > \boldsymbol{0}$, then P_i is regarded as a zero-matrix with $p_i = 1$.

The (n, p_i)-matrix $Q_{i-1} P_i$ contains all the extremal directions of K_i defined by (3), where $A_i^{\mathrm{T}} = [A_{i-1}^{\mathrm{T}}, \boldsymbol{a}_i]$, but redundant vectors may occur in $Q_{i-1} P_i$ and their filtering out is essential. The matrix $Q_{i-1} P_i$ contains the vectors $(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_r) \boldsymbol{x}_t - (\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{e}_t) \boldsymbol{x}_r = (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_r) \boldsymbol{x}_t - (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_t) \boldsymbol{x}_r$ for all $t \in N_-$, $r \in N_+$, which are the cross-points of hyperplane $\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{w} = 0$ and lines connecting all the pairs of extremal directions of K_{i-1} separated by the hyperplane $\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{w} = 0$. Among them, vectors may occur which are not extremal directions of K_i. We must determine for which vectors of $Q_{i-1} P_i = [\boldsymbol{y}_1, \dots, \boldsymbol{y}_p]$ $(p = p_i)$ the corresponding vectors

$$\boldsymbol{z}_k = \left(\boldsymbol{y}_k^{\mathrm{T}}, (-A_i \boldsymbol{y}_k)^{\mathrm{T}} \right)^{\mathrm{T}} \quad (k = 1, \dots, p_i) \tag{4}$$

are the basic solutions of the system

$$\boldsymbol{w}, \quad \boldsymbol{u} \geqslant \boldsymbol{0}, \quad \boldsymbol{1}^{\mathrm{T}} \boldsymbol{w} = 1, \quad A_i \boldsymbol{w} + E_i \boldsymbol{u} = \boldsymbol{0}, \tag{5}$$

where E_i denotes the (i, i)-unit matrix.

Lemma. *Vectors* (6) *are basic solutions of* (5) *for all* $k \in N_0 \cup N_-$, *where*

$$\left(\boldsymbol{x}_k^{\mathrm{T}}, (-A_{i-1} \boldsymbol{x}_k)^{\mathrm{T}}, -\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_k \right)^{\mathrm{T}}. \tag{6}$$

Proof. The matrix of the system of equations is

$$B_i = \begin{bmatrix} A_i & E_i \\ \boldsymbol{1}^{\mathrm{T}} & \boldsymbol{0}^{\mathrm{T}} \end{bmatrix},$$

with rank $i + 1$, so basic solutions of (5) contain at most $i + 1$ positive entries. Since $\boldsymbol{x}_k \in K_{i-1}$ vectors referred by (6) are feasible solutions of (5) for all $k \in N_0 \cup N_-$. The column vectors of B_{i-1} corresponding to the positive entries of (6) are linearly independent, so their extended vectors in B_i are linearly independent of each other, just as they are independent of the vector $(0, \dots, 0, \overset{i}{1}, 0)^{\mathrm{T}}$ (with the '1' on the i-th position) corresponding to the entry $-\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x}_k > 0$ for $k \in N_-$, i.e. (6) are basic solutions of (5) for all $k \in N_0 \cup N_-$. □

It follows from this lemma that it is sufficient to determine for which vectors $\boldsymbol{y}_k$, $k \in \{S + T + 1, \dots, p_i\}$—derived from the multiplication of Q_{i-1} by the submatrix $[\boldsymbol{e}_{t_1 r_1}, \dots, \boldsymbol{e}_{t_T r_R}]$ of P_i—the corresponding $\boldsymbol{z}_k$ has at most $i + 1$ positive entries, and the column vectors of B_i corresponding to these entries are linearly independent.

Having filtered out the redundant vectors from $Q_{i-1} P_i$, we get the (n, q_i)-matrix $(q_i \leqslant p_i) Q_i$, the column vectors of which are the extremal directions of K_i.

If $i < m$ then the $(i + 1)$-th step follows. If $i = m$, then we standardize Q_i, i.e. we require the fulfilment of (1) for each column of Q_i. The centroid of $L = L_m$ is given by

$$\hat{\boldsymbol{w}} = Q_m \boldsymbol{\alpha} \tag{7}$$

where $\boldsymbol{\alpha}$ is the following q_m-dimensional vector

$$\boldsymbol{\alpha}^{\mathrm{T}} = (1/q_m, \dots, 1/q_m). \tag{8}$$

If $\boldsymbol{b}_i < \boldsymbol{0}$, then $L_i = \emptyset$, i.e. the i-th judgement is inconsistent with the preceding ones.

4. An asking algorithm and some characteristics

In the following we outline a possible asking procedure. The fewer the number of questions, the easier to apply the method. On the other hand, the more restricted the feasible set, the more correct the estimation. We propose pairwise comparisons and the asking algorithm of the Churchman–Ackoff method [4], since these provide a proper balance between these expectations.

First, a sequence of the single criteria is determined using pairwise comparisons. Due to the intransitivity of the relation I, we need all the $(n-1)(n-2)/2$ judgements, because a situation bRa, cRb but aPc can occur. The decreasing order of the estimated weights is regarded as the sequence $C_1, \dots, C_{n-1}$ of criteria, but $C_i P C_j$ iff $w_i > w_j + w_n$, $i > j$. Secondly, the decision maker is requested to compare subsets of criteria to a single criterion. In the i-th step ($i = 1, \ldots, n-3$) C_i is compared with $\{C_{i+1}, \ldots, C_j\}$ ($i+2 \leqslant j \leqslant n-1$) for some j, following the algorithm.

Step 1. $i := 1$; $j := n-1$
Step 2. If $C_i P\{C_{i+1}, \ldots, C_j\}$
then go to Step 3
else if $j > i+2$
then $j := j-1$; go to Step 2
else go to Step 3
Step 3. If $i < n-3$
then $i := i+1$; $j := n-1$; go to Step 2
else stop.

This part requires at most $(n-2)/(n-3)/2$, the whole procedure in total at most $(n-2)^2$ comparisons from the decision maker.

In the i-th step we have to store the matrices A_i with (i, n), Q_{i-1} with (n, q_{i-1}), Q_i with (n, q_i) and an $(i+1, i+1)$-matrix to settle the linear independence. It is not necessary to construct P_i, since Q_i can be formed directly from Q_{i-1} and from $\boldsymbol{b}_i^{\mathrm{T}} = \boldsymbol{a}_i^{\mathrm{T}} Q_{i-1}$. An upper bound for q_i is the maximal number of bases, i.e. $\binom{n+i}{n+1}$. For details on the number of vertices, see [7,14].

Every point of L_i is consistent with all the first i judgements. At the end of the i-th step the deviation of the real weights and the centralized weights is upper bounded by

$$R_i = \max\left\{ d\left(\hat{\boldsymbol{w}}^{(i)}, \ \boldsymbol{y}_k\right) : 1 \leqslant k \leqslant q_i \right\}$$

where $Q_i = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q]$ ($q = q_i$) and $d(\cdot,\cdot)$ is defined by

$$d(\boldsymbol{x}, \ \boldsymbol{y}) = \left(\sum_{j=1}^{n-1} \left((\boldsymbol{x})_j - (\boldsymbol{y})_j \right)^2 \right)^{1/2}.$$

It is also expedient to introduce the quantities

$$\beta_i = R_i / \min\left\{ d\left(\hat{\boldsymbol{w}}^{(i)}, \ \boldsymbol{y}_k\right) : 1 \leqslant k \leqslant q_i \right\}$$

to measure the 'regularity' of the polyhedron. The closer this number is to 1, the more the centralized weights are at the centre of the feasible set of weights. Each of these characteristics can easily be derived from the process.

5. Using judgements with different certainties

We can estimate the real weights more precisely if we take into account the vertices to non-equal extents, but in proportion to the certainties of the judgements. It needs further information from the decision maker. If he/she is more infallible in his/her i-th judgement than in the j-th one, then the real weights must be farther from the hyperplane $|\boldsymbol{a}_{w}^{T}| = 0$ than from $\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{w} = 0$. Let the ratio of these distances be λ_i / λ_j, where $i, \ j \in \{1, \ldots, m\}$, $\lambda_i > \lambda_j > 0$ ($i \neq j$) and having finished the process

$$\sum_{i=1}^{m} \lambda_i = 1$$

must hold. We need μ_i-s for which

$$\mu_i / \mu_j = \lambda_j / \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} \mu_i = 1. \tag{9}$$

It can be easily seen that the μ_i-s defined by

$$\mu_i = 1 \Bigg/ \left(\lambda_i \sum_{j=1}^{m} 1/\lambda_j \right) \quad (i = 1, \ldots, m)$$

satisfy (9). We can use them as measures of 'closeness' to the hyperplanes.

We determine the weight of a vertex as the normalized mean of the μ_i-s of the hyperplanes passing through it. Let

$$\chi_{ij} = \begin{cases} 1 & \text{if } (A_m Q_m)_{ij} = 0 \quad (i = 1, \ldots, m; \\ 0 & \text{if } (A_m Q_m)_{ij} < 0 \quad \ \ j = 1, \ldots, q_m). \end{cases}$$

Then

$$\alpha_j' = \left(\sum_{i=1}^{m} \chi_{ij} \mu_i \right) \Bigg/ \left(\sum_{i=1}^{m} \chi_{ij} \right) \quad (j = 1, \ldots, q_m),$$

and for (8) we get

$$\alpha_j = \alpha_j' \Bigg/ \left(\sum_{k=1}^{q_m} \alpha_k' \right)$$

We estimate the real weights through (7).

6. An illustrating example

Let us suppose that we have four criteria ($n-1 = 4$), and the decision maker does not stop the procedure, because reaching satisfactory values. We use the notations of the previous sections.

First, a sequence of the criteria (according to the decreasing order of their estimated weights) is determined using pairwise comparisons. The first four judgements, their representation, the centralized weights $w_1, \ldots, w_4$ and the threshold δ, the quantities R_i, β_i—to characterize the current feasible set of weights—and the maximal value for the threshold Δ_i are the following:

1. C_1PC_2: $\quad -w_1 + w_2 + \delta \leqslant 0,$

$w_1 = 0.5, \quad w_2 = 0.1, \quad w_3 = 0.2, \quad w_4 = 0.2,$

$\delta = 0.2, \quad R_1 = 0.97, \quad \beta_1 = 1.98, \quad \Delta_1 = 1;$

2. $\mathrm{Not}(C_4PC_3)$: $\quad -w_3 + w_4 - \delta \leqslant 0,$

$w_1 = 0.5, \quad w_2 = 0.1, \quad w_3 = 0.3, \quad w_4 = 0.1,$

$\delta = 0.2, \quad R_2 = 0.87, \quad \beta_2 = 1.71, \quad \Delta_2 = 1;$

3. C_1PC_4: $\quad -w_1 + w_4 + \delta \leqslant 0,$

$w_1 = 0.57, \quad w_2 = 0.1, \quad w_3 = 0.26, \quad w_4 = 0.07,$

$\delta = 0.2, \quad R_3 = 0.93, \quad \beta_3 = 2.50, \quad \Delta_3 = 1;$

4. C_2PC_3: $\quad -w_2 + w_3 + \delta \leqslant 0,$

$w_1 = 0.58, \quad w_2 = 0.27, \quad w_3 = 0.11, \quad w_4 = 0.04,$

$\delta = 0.07, \quad R_4 = 0.51, \quad \beta_4 = 3.31, \quad \Delta = 0.33.$

The matrix of inequalities and the vertices of the current feasible set (contained the threshold values in the lowest row) are

$$A_r = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{2}{5} \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{5} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

We can get all the weights and thresholds consistent with these judgements as convex combination of the columns of Q_4. We propose the mean values as a compromise among the immoderate selecting strategies, the result of which are represented by the column vectors. The fifth judgement

5. C_1PC_3: $\quad -w_1 + w_3 + \delta \leqslant 0$

is redundant, because the decision maker was consistent with the transitivity of P, hence $Q_5 = Q_4$. The reverse judgement would make the feasible set empty. Since each column in Q_5 consistent with

6. C_2PC_4: $\quad -w_2 + w_4 + \delta \leqslant 0,$

therefore after the pairwise comparisons we have the sequence C_1, C_2, C_3, C_4 for the subsequent comparisons, but this does not mean a ranking in P.

In the 7-th step we get

7. $\mathrm{Not}(C_1P\{C_2, C_3, C_4\})$:
$w_1 - w_2 - w_3 - w_4 - \delta \leqslant 0,$

and after having filtered out the redundant vectors we have

$$Q_7 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{2}{5} & \frac{2}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{5} & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix}, \quad \begin{aligned} w_1 &= 0.48, \\ w_2 &= 0.32, \\ w_3 &= 0.16, \\ w_4 &= 0.04, \\ \delta &= 0.07, \end{aligned}$$

$R_7 = 0.25, \quad \beta_7 = 1.95, \quad \Delta_7 = 0.33.$

The result of next comparison

8. $C_1P\{C_2, C_3\}$: $\quad -w_1 + w_2 + w_3 + \delta \leqslant 0$

excludes only the second column from Q_7 and no new vector arises, so the values change slightly:

$w_1 = 0.52, \quad w_2 = 0.32, \quad w_3 = 0.11, \quad w_4 = 0.05$

$\delta = 0.08, \quad R_8 = 0.24, \quad \beta_8 = 1.48, \quad \Delta_8 = 0.33.$

From the last comparison

9. $\mathrm{Not}(C_2P\{C_3, C_4\})$: $\quad w_2 - w_3 - w_4 - \delta \leqslant 0$

we have for the vertices of the final feasible set and for the centralized weights and threshold

$$Q_9 = \begin{bmatrix} \frac{2}{3} & \frac{2}{5} & \frac{2}{4} & \frac{3}{7} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{4} & \frac{2}{7} \\ 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{7} \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} w_1 &= 0.50, \\ w_2 &= 0.27, \\ w_3 &= 0.15, \\ w_4 &= 0.08, \\ \delta &= 0.08, \end{aligned}$$

$R_9 = 0.25, \quad \beta_9 = 2.69, \quad \Delta_9 = 0.33.$

If the decision maker is satisfied with these values, we can draw the conclusion that $\{C_1, C_4\}P\{C_2, C_3\}$, because $0.50 + 0.08 > 0.27 + 0.18 + 0.08$ holds for their weights.

References

[1] Bouyssou, D., and Vansnick, J.C., "Noncompensatory and generalized noncompensatory preference structures", Cahier du LAMSADE, Universite du Paris-Dauphine, 1985.

[2] Brans, J.P., Mareschal, B., and Vincke, Ph., "PROMETHEE: a new family of outranking methods in multicriteria analysis", in: J.P. Brans, (ed.), *Operational Research '84*, North-Holland, Amsterdam, 1985.

[3] Brans, J.P., and Vincke, Ph., "A preference ranking organization method, the PROMETHEE method", *Management Science* 31 (1985) 647–656.

[4] Churchman, C.W., Ackoff, R.L., and Arnoff, E.I., *Introduction to Operations Research*, Wiley, New York, 1957.

[5] Fishburn, P.C., *Utility Theory for Decision Making*, Wiley, New York, 1970.

[6] Jacquet-Lagrèze, E., and Siskos, J., "Assessing a set of additive utility functions for multicriteria decision-making, the UTA method", *European Journal of Operational Research* 10 (1982) 151–164.

[7] Klee, V., "On the number of vertices of a convex polytope", *Canadian Journal of Mathematics* 16 (1964) 701–720.

[8] Knoll, A.L., and Engelberg, A., "Weighting multiple objectives—The Churchman–Ackoff technique revisited", *Computers and Operations Research* 5 (1978) 165–177.

[9] Pekelman, D., and Sen, S.K., "Mathematical programming models for the determination of attribute weights", *Management Science* 20 (1974) 1217–1229.

[10] Pfanzagl, J., *Theory of Measurement*, Physica, Würzburg, 1968.

[11] Roy, B., "Classement et choix en presence de points de vue multiples (la methode ELECTRE)", *Revue Française d'Automatique, Informatique et Recherche Operationelle* 8 (1968) 57–75.

[12] Roy, B., and Bertier, P., "La methode ELECTRE II, une application au media-planning", in: M. Ross (ed.), *Operational Research '72*, North-Holland, Amsterdam, 1973.

[13] Roy, B., "ELECTRE III: Un algorithme de classement fondé sur une representation floue des preferences en presence de criteres multiples", Cahiers du Centre Etudes Recherche Operationelle 20 (1978) 3–24.

[14] Saaty, T.L., "The number of vertices of a polyhedron", *The American Mathematical Monthly* 62 (1955) 326–331.

[15] Srinivasan, V., Shocker, A.D., "Estimating the weights for multiple attributes in a composite criterion using pairwise judgments", *Psychometrika* 38 (1973) 473–493.

[16] Vansnick, J.C., "Strength of preference: Theoretical and practical aspects", in: J.P. Brans (ed.), *Operational Research '84*, North-Holland, Amsterdam, 1985.

[17] Vansnick, J.C., "On the problem of weights in multiple criteria decision-making (the noncompensatory approach)", *Cahier du LAMSADE* 57, Universite du Paris-Dauphine, 1985.