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COMMENTS ON THE γ-MODEL

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Recently, a class of operators has been suggested, called γ-model, which is equipped with several suitable formal properties and additionally satisfies empirical aspirations. First, its formal development from a more abstract point of view is presented. Second, a dual form and some possible interpretations are offered. Third, it is shown how to evaluate the classificatory power of each operator within the γ-model.

1. INTRODUCTION

If it is intended to design an operator for the connection of fuzzy truth values or for the aggregation of fuzzy sets, it is reasonable to require mathematical properties such as continuity, strict monotonicity, injectivity (implied by continuity and strict monotonicity), commutativity, and associativity. Commutativity is given if the operator is injective and associative [1]. Moreover, it should be in accordance with the truth tables of dual logic. Several operators have been suggested which satisfy these requirements: Minimum/Maximum [7], Product/Algebraic Sum [2], Hamacher's $D(x,y)/K(x,y)$ [5], Yager's $C_p(x)/D_p(x)$ [6], and others.

It has been shown [3] that operators with the above properties result in membership grades between zero and minimum in the case of intersection (and), and between maximum and one in the case of union (or), respectively:

$$C(\mu_A(x),\mu_B(x)) = \min\ (\mu_A(x),\mu_B(x)) \quad (1)$$
$$D(\mu_A(x),\mu_B(x)) = \max\ (\mu_A(x),\mu_B(x)) \quad (2)$$

Minimum and maximum can be obtained by the limes of a series of operators.

However, empirical investigations [8, 10] have shown that human aggregation usually provides membership values between minimum and maximum. Presumably, operators based on the above properties are too restrictive for judgmental behaviour. In order to design an averaging model, at least one of the assumptions will have to be weakened. The slightest loss of generality seems to be entailed by the abandonment of associativity, as the only averaging operator satisfying this property is the median [4].

Recently, a class of operators [9] has been suggested which is equipped with the above formal properties (except associativity) and additionally fulfils empirical aspirations [8]:

$$\mu_\gamma = (\Pi_{\mu_i}(x)^{\delta_i})^{1-\gamma}(1-\Pi(1-\mu_i(x))^{\delta_i})^\gamma \qquad \begin{matrix} 0\le\mu(x)\le 1 \\ 0<\delta\le 1 \\ 0<\gamma\le 1 \end{matrix} \quad (3)$$

Starting from the convex combination [7] of an intersecting and a unifying operator, its modelling quality within a theory of concept integration has been examined [10]. Our first intention is to present its formal development from a more abstract point of view. A second notion concerns the dual form of the γ-model and some possible interpretations.

2. The general structure of the γ-model

Let X be the universe of discourse with the elements x. A,B and Γ are fuzzy sets in X. Then, the convex combination of A,B and Γ can be denoted by (A,B;Γ) and is defined by the relation

$$(A,B;\Gamma) = \overline{\Gamma}A + \Gamma B \tag{4}$$

where $\overline{\Gamma}$ is the complement of Γ. Written in terms of membership functions, (4) reads

$$f_{(A,B;\Gamma)} = (1-f_\Gamma(x))\cdot f_A(x)+f_\Gamma(x)\cdot f_B(x) \tag{5}$$

A basic property of the above-defined convex combination is expressed by (6):

$$A \cap B \subset (A,B;\Gamma) \subset A \cup B \tag{6}$$

Obviously, the convex combination is a partial fuzzy set between the intersection and the union of two fuzzy sets A and B.

One easily imagines A and B being themselves results of set operations. For reasons of clearness A is replaced by $L_\cap$ and B by $L_\cup$: A = = $L_\cup$, B = $L_\cup$. Defining

$$L_\cap = C \cap D \tag{7}$$

$$L_\cup = C \cup D \tag{8}$$

(6) can be rewritten:

$$L_\cap \cap L_\cup \subset (L_\cap, L_\cup;\Gamma) \subset L_\cap \cup L_\cup \tag{9}$$

As $L_\cap \subset L_\cap \cap L_\cup$ and $L_\cup \supset L_\cap \cup L_\cup$, relationship (9) can be simplified:

$$L_\cap \subset (L_\cap, L_\cup;\Gamma) \subset L_\cup \tag{10}$$

The convex combination of the intersection and the union of several sets is a partial fuzzy set between intersection and union.

The convex combination of $L_\cap$, $L_\cup$ and Γ can now be defined similarly to (4). In order to represent relation (9) by a general algebraic form, we continue the initial operator notation instead of membership terms:

$$\Theta_{\gamma(x)}(\mu_A(x),\mu_B(x)) = (1-\gamma(x))C(\mu_A(x),\mu_B(x))+ +\gamma(x)D(\mu_A(x),\mu_B(x)) \tag{11}$$

If it is assumed that the result of the convex combination merely depends on the basic operations C and D, then the variable $\gamma(x)$ becomes constant:

$$\Theta_\gamma(\mu_A(x),\mu_B(x)) = (1-\gamma)C(\mu_A(x),\mu_B(x))+ + D(\mu_A(x),\mu_B(x)) \tag{12}$$

Formula (12) represents the general form of the convex combination.

If C and D are continuous, strictly monotonic, and associative, as demanded in the beginning, then their convex combination is continuous, strictly monotonic, injective, and commutative:

- If C and D are continuous, then their sum will be continuous, too.
- If C and D are strictly monotonic, then their sum will be strictly monotonic, too.
- If the convex combination is continuous and strictly monotonic, then it is injective.
- The commutativity is given as

$$\begin{aligned}\Theta(\mu_A(x),\mu_B(x)) &= (1-\gamma)\cdot C(\mu_A(x),\mu_B(x)) + \\ &+ \gamma\cdot D(\mu_A(x),\mu_B(x)) = \\ &= (1-\gamma)\cdot C(\mu_B(x),\mu_A(x)) + \\ &+ \gamma\cdot D(\mu_B(x),\mu_A(x)) = \\ &= \Theta(\mu_B(x),\mu_A(x))\end{aligned} \tag{13}$$

Now, in order to get a larger scope for applications and empirical research we will admit continuous and monotonic transformations on the elementary operations given in (12):

$$\begin{aligned}\mu_\gamma(x) &= f(\Theta(\mu_A(x),\mu_B(x)) = \\ &= (1-\gamma)\cdot f(C(\mu_A(x),\mu_B(x)) + \\ &+ \gamma f((D(\mu_A(x),\mu_B(x)))\end{aligned} \tag{14}$$

The transformation can be chosen with respect to scaling aspects, psychological interpretations, theoretical conditions, empirical facts, modelling interests, and so forth. Formula (14) is the general γ-model.

3. The dual form

The most simple specification is obtained by representing the conjunction by the product and the disjunction by the algebraic sum. These two operators are the only polynomial solutions if the above axioms are satisfied.

If, for instance, in view of a psychological interpretation [10], the transformation is defined by the logarithm, then the result will be the γ-model given by equation (3). On the other hand, its dual form can be derived by applying the transformation log(1-x):

$$\mu_{\gamma'}(x) = 1-(1-\prod_i^m \mu_i(x))^{1-\gamma}(\prod_i^m(1-\mu_i(x)))^{\gamma} \quad \begin{matrix} 0\le\gamma<1 \\ 0\le\mu(x)\le 1 \end{matrix} \tag{15}$$

Models (3) and (15) are both in accordance with the truth tables of dual logic. A more differentiated aggregation can be provided by introducing weights δ_i with $\sum_{i=1}^m \delta_i = m$. If the negation of $\mu(x)$ is $1-\mu(x)$ then the de Morgan rules are satisfied:

$$\begin{aligned}\mu_{\gamma'}(x) &= 1-(1-\prod\mu_i(x))^{1-\gamma}(1-(1-\prod(1-\mu_i(x))))^{\gamma} \\ &= 1-(1-\prod\mu_i(x))^{1-\gamma}(\prod(1-\mu_i(x)))^{\gamma}\end{aligned} \tag{16}$$

Comparing the values $\mu_\gamma(x)$ and $\mu_{\gamma'}(x)$, it can be stated that

$$\mu_\gamma(x) \le \mu_{\gamma'}(x) \tag{17}$$

The γ-model has already been studied empirically [8, 10] within the framework of evaluation theory. An application together with its dual form might, for instance, be found in a decision-making typology of optimizers and satisfiers. While the first would need, say, a car which is "quick and comfortable", the second would accept a vehicle which is "not slow and not uncomfortable". Representing the optimizer's judgmental concept by the primal and the satisfier's judgmental concept by the dual form, generally higher evaluations are predicted for the latter with respect to the specified list of criteria.

4. Aspects of classification

Besides its role as a class of operators for the aggregation of fuzzy sets, the γ-model may be used if the grade of impreciseness of a certain operator with respect to classification is of importance.

In crisp set theory two sets X and Y induce a classification on X if the characteristic functions

$$\chi_{X \cup Y} = 1 \tag{18}$$
$$\chi_{X \cap Y} = 0 \tag{19}$$

are satisfied. Moreover, if $Y = \overline{X}$ then

$$\chi_Y = 1-\chi_X \tag{20}$$

In fuzzy set theory the characteristic function is replaced by the membership function, i.e. (18) and (19) are not satisfied. However, one may evaluate the grade of classificatory power c associated with a certain operator by the integral of the aggregated functions $\mu_A(x)$ and $\mu_{\overline{A}}(x)$:

$$c_\cap = \int_0^1 \mu_{A\cap\overline{A}}(x)\,dx \tag{21}$$
$$c_\cup = \int_0^1 (1-\mu_{A\cup\overline{A}}(x))\,dx \tag{22}$$

For $0 < \gamma < 1$ this measure is monotonic and takes the values of the interval [0,1], perfect classification being indicated by zero and indiscriminability by one. As an example, the classificatory power of Minimum and Maximum may be considered. The corresponding membership values for the intersection and the union are given by

$$\mu_{A\cap\overline{A}}(x) = \min\,(\mu_A(x),\mu_{\overline{A}}(x)) \tag{23}$$
$$\mu_{A\cup\overline{A}}(x) = \max\,(\mu_A(x),\mu_{\overline{A}}(x)) \tag{24}$$

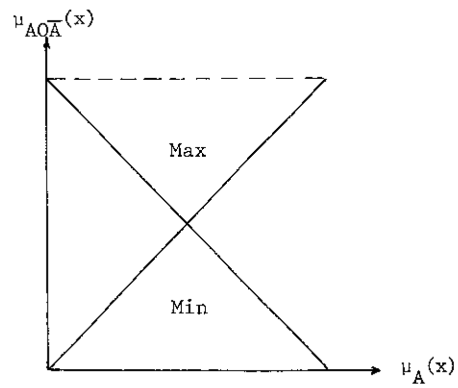


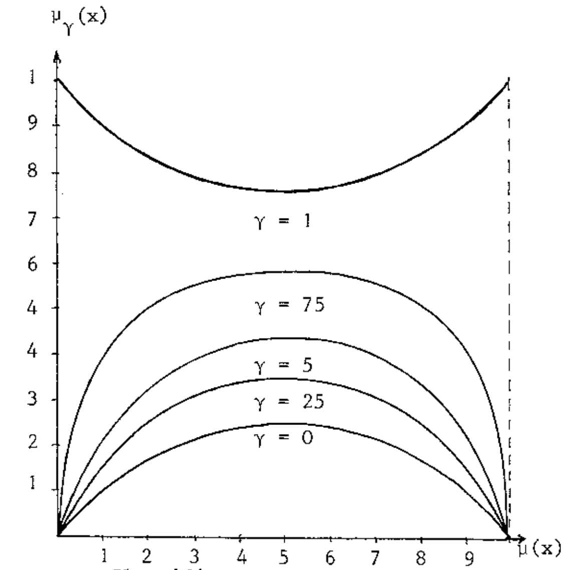
Fig. 1: Classificatory power of Minimum and Maximum

Figure 1 shows the corresponding graphs of the membership functions. The values of the respective definite integrals are obviously $c_{min} = 1/4$ and $c_{max} = 1/4$.

The classificatory power of the γ-model is not fixed, since $c_\Theta = f(\gamma)$ (Figure 2 and 3). The -model equals the product and the algebraic sum for $\gamma = 0$ and $\gamma = 1$, respectively. By solving the corresponding definite integrals

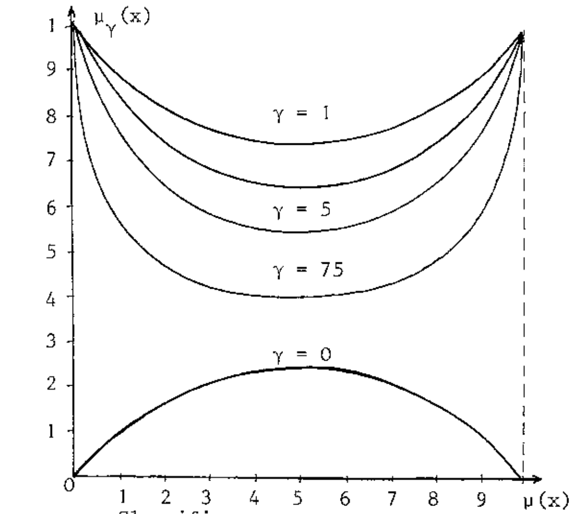
$\int_0^1 x(1-x)\,dx$ and $\int_0^1 x+(1-x)-x(1-x)\,dx$, it is easy to verify that

$$1/6 < c < 5/6 \qquad (24)$$



Classificatory power
Fig. 2. of the primal form of the γ-model

Classificatory aspectc are of importance in most conceptual systems in technical, economical, social, medical, and other areas. Very often, the used concepts do not make up disjunctive classes. Technical and biological systems usually dispose of partially compensatory subsystems in order to overcome local feebleness. Hence, the utility of models and consequently their practicability will be increased if fuzzy classes and operators are used in representing a given situation, for instance for simulations, cybernetic systems, evaluative hierarchies, and others.



Classificatory power
Fig. 3. of the dual form of the γ-model

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