The problem of weighting

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Abstract

Using the structural form supplied many valued logic (and fuzzy set theory) a new general method of solving multiple aspect decision problems is presented. This method has the advantage of being able to solve multiple objection decision making depending on the applied operator. Our aim is to consider the connection between the evaluation, the weighting and the aggregation.

1 The Interpretation of weighting

If we examine the heuristically based decision procedures (including those modeled with the aid of fuzzy sets), it may be seen that the values according to the individual criteria are taken into account in a changed (weighted) manner in the course of the aggregation. However, the connection between the values, the weights and the aggregation has not been sufficiently clarified. This can be described to the fact that very many types of weighting procedures have arisen.

Saaty first investigated the method of determination of weights measurable on a proportional scale. The problem of the connection with the aggregation does not arise here. Various proposals are to be found in the papers of Yager concerning the theory of fuzzy sets; we shall return to these later. Our aim below is to consider the connection between the evaluation, the weighting and the aggregation. Naturally, we shall deal first with the interpretation of weighting.

The interpretation of weighting will be illustrated on two examples, neither of which relates to the theory of multi-criteria decisions in the strict sense. These examples are taken from the areas of pattern recognition (recognition of letters) and medical diagnosis. The syntactic pattern recognition method breaks down the pattern into its elements, these elements are compared with primitives, and the letters are then identified by the description of the structural correlations of these elements. The types of the primitives and the elements of the resolution are not the same. Thus, to each element we may ascribe not only the primitive class, but also the degree of belonging to the class in question (the extent of similarity and adequacy). As an example, let us consider three letters: a, o and u. Each of these three written letters has a tail in a certain position, which can be represented by primitive classes. The belonging of a letter to be recognized in the letter a class is determined primarily by the adequacy concerning the tail primitives, independently of the adequacy relating to the other primitives, if we wish to discriminate this from the class of the letter o; however, of discrimination is made from the letter u, the extent of closedness from above will be the more determining. Accordingly, the existence and position of certain elements are of importance, while those of others are irrelevant as regards the discrimination class. As an example, the closedness from below does not play a role in the decision here.

The recognition and identification is thus the result of two types of evaluation:

(a) the extent of the adequacy (goodness), and this is ascribed to the object;

(b) the extent of the importance (weight) of the individual elements, and this is ascribed to the class to be recognized.

As we have seen, the values are re-evaluated from the aspect of the letter a, and with their use the information necessary for the decision will already be appropriate. We have also seen that the importance must characterize not only the individual class, but also the set of classes to be discriminated, for only in this way can the classes be differentiated, i.e. these classes should form a "classification" as regards the set of objects.

(a) Objects

$$\begin{array}{ccc} o_1, & o_2, ..., & o_n \\ x_{11}, & x_{21}, ..., & x_{n1} \\ x_{12}, & x_{22}, ..., & x_{n2} \\ & \vdots & \\ x_{1l}, & x_{2l}, ..., & x_{nl} \end{array} \quad (1)$$

(b) Classes

$$\begin{array}{ccc} A_1, & A_2, ..., & A_k \\ w_1^{(1)} & w_1^{(2)}, ..., & w_1^{(k)} \\ w_2^{(1)} & w_2^{(2)} & w_2^{(k)} \\ & \vdots & \\ w_l^{(1)} & w_l^{(2)} & w_l^{(k)} \end{array} \quad (2)$$

Although this has not been emphasized, the extent of adequacy plays determining role too. Thus, the process of classification is a function of the extent of closedness from above.

The recognition and identification can be achieved as follows with regard to an object o_i: Let us assume that o_i belongs to class A_j. The degree of o_i belonging in A_j is determined with the aid of the structural description of o_i and the values $x_{i1}^{(1)}, ..., x_{il}^{(1)}$ produced by consideration of the importance weights

$w_1^{(1)}, w_2, ..., w_l^{(1)}$ and the evaluations $x_{i1}, ..., x_{il}$. If the procedure is carried out for classes $j = 1, ..., k$, a decision function covering the alternatives may be obtained. By means of the examination of the decision function, not only the class but also the definition (fuzziness) of the decision can be ascribed to our decision. This measure will be defined exactly below. However, the following figure illustrates what we mean by sharpness:

In the giving of a medical diagnosis to a patient we are faced with a similar task. The patient's symptoms involve (preliminary) evaluation. This can be interpreted on the basis of the former example, as the values ascribed to the objects. From the aspect of the individual possible diagnosis classes, these symptoms may be important, but they may also be irrelevant. (For instance, a rise in temperature may be evaluated in different ways) For the full evaluation, therefore, there is a need for the values ascribed to the classes, and the weights must be given. Naturally, besides these evaluations, the logical structure of the individual diagnosis classes that is based on the evaluations has determining role too. If our task is merely the study of a single class, i.e. an object belongs to the class in question or not, then the separation of the values and weights is not necessary. The values combined in this way (referred to below as transformed values), however, are unsuitable for the differentiation of disease classes, for they are bound to a hypothesis and are thus not "objective". Concrete psychological examinations performed to map the character of the decision support the above assumption. The physician assumes a disease and questions the patient from this aspect; if he cannot confirm his hypothesis satisfactorily, he puts questions from the aspect of the hypothesis of another diagnosis, and may even repeat certain questing. Here, two behavioral forms are possible: alignment of the supporting arguments or of the counter-indications corresponding to the hypothesis.

In the case of multi-criteria decisions, the various objects may designate various sets of aims. (For the buying of clothes or a radio, evaluation is made according to other classes of aims.) If we now evaluate three objects (a, b and c), by comparing them in pairs, then the decision is taken at the intersection of the sets of the aims (aspects) (i.e. in accordance with the common aims), and the elements of the section are different if the pairs a, b; b, c; and a, c are compared. For example, therefore, a can be evaluated in accordance with aims c_1 and c_3, b in accordance with c_1 and c_2 and c in accordance with c_2 and c_3; thus, it can be said that in the cases of object a, b and c, c_2, c_3 and c_1, respectively are negligible and of no importance; (a, b) may then be compared according to $c_1, (b, c)$ to c_2, and (a, c) to c_3, and hence the three objects may form an inconsistent cyclic trio.

After this digression, let us examine the connection between the decision procedure, the values and the importance.

2 Character of importance transformations of values

We have seen that the values are transformed in accordance with the importance. This is denoted in the following way:

$$x_i = R\left(w_i, x_i\right) \tag{3}$$

It might be thought that a natural restriction on the value transformation is that it is a monotonously increasing function of the importance. That is, if a value is more important, then a larger value must be taken into consideration in the aggregation, and vice versa. The false nature of this conception is well illustrated by an example relating to the usual decision procedure in fuzzy sets. Let the extent of the degree of existence of two properties interpreted on a set of alternatives be $\mu_A\left(x_i\right)$ and $\mu_B\left(x_i\right)$ respectively. The fuzzy decision procedure designates the alternative for which

$$\min\left(\mu_A\left(x_i\right)\mu_B\left(x_i\right)\right) \tag{4}$$

is maximum. That is, with every alternative characterized by its poorest property, the maximum one is selected. (See figure)

Let us denote this alternative by x_D

Figure 2.

Let us assume that property $\mu_A\left(x\right)$ is of greater importance than $\mu_B\left(x\right)$. If the value of $\mu_A\left(x\right)$ is accordingly increased and that of $\mu_B\left(x\right)$ is decreased, the alternative x_D obtained by repeated performance of the decision procedure lies closer to the maximum of $\mu_B\left(x\right)$, though because of the more important nature of $\mu_A\left(x\right)$, x_D should move in the direction of the maximum of $\mu_A\left(x\right)$. Thus, the desired effect may be attained with a change that is just the opposite in character.

Figure 3.

In the application of the min. operator, the transformation $R\left(w, x\right)$ must possess the following properties, where it is assumed that $w \in [0, 1]$ and if their properties are equally important then $w = 1$, i.e.

$$R\left(1, x\right) = x \tag{5}$$

If something is not of importance (i.e. w assumes the value 0), then it does not play a role in the decision only if

$$R\left(0, x\right) = 1 \tag{6}$$

If $x = 1$ and $w = 0$, then as we have seen $R\left(0, 1\right) = 1$ and clearly $R\left(1, 1\right) = 1$; thus, let $R\left(w, 1\right) = 1$. We have not yet examined the nature of $R\left(w, 0\right)$. If some property of an alternative is missing, then one possible principle is that, independently of its importance, this alternative should feature with value 0 (i.e. a necessary condition is the existence of the property).

Since $R(0,x)=1$ and

$$R(w,0)=0, \tag{7}$$

$R(w,x)$ is not continuous at the point $(0,0)$.

Another possible principle is that the transformed value should increase with decrease of the importance.

$$R(w,0)=w \tag{8}$$

Thus, it should be of an ever smaller role in the decision. This is in accordance with the more general condition that

$$\text{if } w_1 \leq w_2, \text{ then } R(w_1,x) \geq R(w_2,x) \tag{9}$$

It is similarly a natural restriction that

$$\text{if } x_1 \leq x_2, \text{ then } R(w,x_1) \leq R(w,x_2) \tag{10}$$

It should be noted that the $R(w,x)$ satisfying the above requirements has a property such as a many-valued logical implication, i.e.

$$x^* = R(w,x) = w \longrightarrow x \tag{11}$$

From other considerations, Yager proposed

$$x = \max(1-w,x) \tag{12}$$

which is a possible definition of the many-valued implications. Other examinations relating to many-valued implications may be found in

Figure 4.

If $R(w,0)=0$, one possible means of importance transformation satisfying the condition is

$$x = R(w,x) = x^w \tag{13}$$

which is to be found in a number of papers.

For a clear picture of the dependence of the importance transformation on the character of the aggregation, let us examine the case when the alternatives are characterized by their best properties and we select the best element:

$$\max_x \max(\mu_A(x), \mu_B(x)) \tag{14}$$

Then, for the important property to determine the decision, the value of $\mu_A(x)$ must be increased in proportion to the importance, while that of the less important one must be decreased. Hence, it can be seen that appropriate demands are

$$R(0,x)=0 \tag{15}$$

$$R(w,0)=0 \tag{16}$$

$$R(1, x) = x \tag{17}$$

Here it is also practical to adopt the considerations relating to the following monotonicities

$$\text{if } w_1 \leq w_2, \text{ then } R(w_1, x) \leq R(w_2, x) \tag{18}$$

$$\text{if } x_1 \leq x_2, \text{ then } R(w, x_1) \leq R(w, x_2) \tag{19}$$

These conditions are not inconsistent with

$$R(w, 1) = w \tag{20}$$

which is likewise assumed.

As an example, the above properties are possessed by the relation

$$x = R(w, x) = \min(w, x) \tag{21}$$

Figure 5.

In the evaluation theory we can distinguish between three types of operators: conjunction, disjunction and aggregation operators. The decision procedure interpreted on the fuzzy sets means the application of the conjunction operator. The other decision procedure featuring in the paper uses the disjunction operator. Let us examine what importance transformation should be employed in the application of the aggregation operator. It is known that, for the aggregation operator, the neutral value divides the interval of evaluation into two parts: positive and negative. Because of the property of the aggregation that $a(x, \nu) = x$ (i.e. the neutral value causes no change in the aggregated value), and thus for that property of the object which is not of importance ($w = 0$), let

$$R(0, x) = \nu \tag{22}$$

Further, it is a natural restriction that in the positive interval the transformed value should increase with increase of the importance, while in the negative this value should decrease with increase of the importance. One such transformation is

$$R(w, x) = \begin{cases} x + \nu(1 - w) & \text{if } w \geq x \\ w + \nu(1 - x) & \text{if } w \leq x \end{cases} \tag{23}$$

$$R(w, \nu) = \nu + w(x - \nu)$$

$R(w, x)$ can be seen in figure 6.

Figure 6.

This figure reveals that we obtain the importance transformation relating to the conjunction operator or to the disjunction operator or to the disjunction operator in the cases $\nu = 1$ and $\nu = 0$, respectively. The importance transformations constructed in this way have the following advantages:

(a) the importance transformations relating to the various operators can be treated in a uniform manner;

(b) since the transformation is linear, for the individual intervals the values may be calculated simply;

(c) the operator properties $c(0, x) = 0$ and $d(1, x) = 1$ hold only in cases with importance $w = 1$.

However, they also have disadvantages:

(a) not only the proposed transformation satisfies the conditions;

(b) the method gives no indication at all as to the determination of the w values, and gives no conditions at all, with the exception of $w_i \in [0, 1]$ (e.g. $\sum_{i=1}^{n} w_i = ?$)

(c) the connection of the aggregation and the importance transformation is only quasi-determined. With the aim of the elimination of these drawbacks, we must turn to a more abstract method of treatment.

3 Condition system relating to weighting (The axioms)

Below we shall omit the condition $w_i \in [0, 1]$ and introduce some symbols merely to facilitate understanding. Thus, let

$$R(w, x) = w \Box x \quad \text{and} \quad a(x, y) = x \diamond y$$

where $a(x, y)$ may mean a conjunction or an aggregation operator.
On the basis of the above considerations, let

1/a

$$1 \Box x = x \tag{24}$$

1/b

$$0 \Box x = \nu \tag{25}$$

1/c

$$w \Box \nu = \nu \tag{26}$$

If $w \geq 0$, let us make some assumptions for the monotonicity of the importance transformation:

2/a

$$\text{if } \nu < x \text{ and } w_1 \leq w_2, \text{ then } w_1 \Box x \leq w_2 \Box x \tag{27}$$

2/b

$$\text{if } \nu > x \text{ and } w_1 \leq w_2, \text{ then } w_1 \Box x \geq w_2 \Box x \tag{28}$$

2/c

$$\text{if } x_1 \leq x_2, \text{ then } w\Box x_1 \leq w\Box x_2 \tag{29}$$

where $\nu = 1$ in the case of a conjunction operator and $\nu = 0$ in the case of a disjunction operator.

It is advisable to demand that the following connections hold between the operator and the importance transformation:

3/a

$$w\Box(x_1 \diamond x_2) = (w\Box x_1) \diamond (w\Box x_2) \tag{30}$$

3/b

$$(w_1\Box x) \diamond (w_2\Box x) = (w_1 + w_2)\Box x \tag{31}$$

3/c

$$w_1\Box(w_2\Box x) = (w_1 w_2)\Box x \tag{32}$$

Since one negation ($\overline{x}$) can always be ascribed to the aggregation operator, for this negation and only for the aggregation operator let us assume that

$$-w\Box x = w\Box\overline{x} \tag{33}$$

Let the generator function of the operator be $f(x)$.
That is

$$x_1 \diamond x_2 = f\left(f^{-1}(x) + f^{-1}(x_2)\right) \tag{34}$$

Proposition 1 *Conditions 1-4 are satisfied by the importance transformation*

$$w\Box x = f^{-1}(wf(x))$$

$$\underline{x} = R(w, x) = w\Box x = f\left(\underline{w}\left(f^{-1}(x)\right)\right) \tag{35}$$

$$(w_1\Box x_1) \wedge (w_2\Box x_2) = f^{-1}(w_1 f(x_1) f(x_2))$$

Proof. 1/a

$$1\Box x = f\left(1f^{-1}(x)\right) = x$$

1/b

$$0\Box x = f\left(0f^{-1}(x)\right) = f(0) = \nu$$

1/c

$$w\Box\nu = f\left(wf^{-1}(\nu)\right) = f(0) = \nu$$

Holding the monotonicity under point 2 is seen first for the aggregation operator:
2/a If $\nu \leq x$,then $f^{-1}(x) \geq 0$, and thus if $w_1 \leq w_2$

$$w_1 f^{-1}(x) \leq w_2 f^{-1}(x)$$

Because of the increasing nature of the monotonicity $f(x)$

$$f\left(w_1 f^{-1}(x)\right) \leq f\left(w_2 f^{-1}(x)\right)$$

2/b If $\nu \geq x$, then the proposition arises with the use of $f^{-1}(x) \leq 0$.

In a conjunction case, $\nu = 1$. Thus, only the case $\nu \geq x$ need be considered. Utilizing the fact that then $f^{-1}(x) \geq 0$ and $f(x)$ is strictly monotonously decreasing, it emerges for $w_1 \leq w_2$ that

$$f\left(w_1 f^{-1}(x)\right) \geq f\left(w_2 f^{-1}(x)\right)$$

In the disjunction case $\nu = 0$, and thus only the case $\nu \leq x$ must be proved. It is also $f^{-1}(x) \geq 0$ here, but $f(x)$ is strictly monotonously increasing, and hence the statement arises similarly as above.

The generator function of the aggregation operator is monotonously increasing. Thus, if $x_1 \leq x_2$, then $wf^{-1}(x_1) \leq wf^{-1}(x_2)$. Using the monotonously increasing nature of $f(x)$ again:

$$f\left(wf^{-1}(x_1)\right) \leq f\left(wf^{-1}(x_2)\right)$$

The disjunctive operator is also monotonously increasing and thus the proof may be performed similarly.

The following statements hold:

a)

$$w \diamond 1 = 1 \quad w \vee 1 = 1 \tag{36}$$

b)

$$w \diamond 0 = 0 \quad w \wedge 0 = 0 \tag{37}$$

The statements may be seen simply with the aid of the properties of the generator functions. Hence, weighting does not eliminate the screening (conjunctive) or highlighting (disjunctive) nature of the logical operators.

In the case of a conjunction operator, if $x_1 \leq x_2$, then $wf^{-1}(x_1) \geq wf^{-1}(x_2)$ since $f^{-1}(x)$ is monotonously decreasing, and thus

$$f\left(wf^{-1}(x_1)\right) \leq f\left(wf^{-1}(x_2)\right)$$

The proof of the properties 3/a, 3/b, and 3/c is performed in the same way for all three operators.

3/a

$$\begin{aligned}
w\Box(x_1 \diamond x_2) &= f\left(wf^{-1}\left(f\left(f^{-1}(x_1) + f^{-1}(x_2)\right)\right)\right) = \\
&= f\left(wf^{-1}(x_1) + wf^{-1}(x_2)\right) = \\
&= f\left(f^{-1}\left(f\left(wf^{-1}(x_1)\right) + f^{-1} f\left(wf^{-1}(x_2)\right)\right)\right) = \\
&= (w\Box x_1) \diamond (w\Box x_2)
\end{aligned}$$

3/b

$$\begin{aligned}(w_1\Box x)\diamond(w_2\Box x) &= f\left(f^{\,-1}\left(f\left(w_1 f^{\,-1}(x)\right)+f^{\,-1}\left(f\left(w_2 f^{\,-1}(x)\right)\right)\right)\right)= \\ &= f\left(w_1 f^{\,-1}(x)+w_2 f^{\,-1}(x)\right)= \\ &= f\left((w_1+w_2)\,f^{\,-1}(x)\right)= \\ &= (w_1+w_2)\,\Box x\end{aligned}$$

3/c

$$\begin{aligned}w_1\Box(w_2\Box x) &= f\left(w_1 f^{\,-1}\left(f\left(w_2 f^{\,-1}(x)\right)\right)\right)= \\ &= f\left(w_1 w_2 f^{\,-1}(x)\right)= \\ &= (w_1 w_2)\,\Box x\end{aligned}$$

Property 4 must be proved only for the aggregation operator, and then for the negation ascribed to the operator it holds that $\overline{x}=f\left(-f^{\,-1}(x)\right)$. Thus:

$$-w\Box x=f\left(-wf^{\,-1}(x)\right)=f\left(wf^{\,-1}\left(f\left(-f^{\,-1}(x)\right)\right)\right)=w\Box\overline{x}$$

■

4 General and special forms of weighting

For construction of the general forms of the operators, it holds that

$$a(x_1,...,x_n)=f\left(\sum_{i=1}^{n} f^{\,-1}(x_i)\right) \tag{38}$$

while for the aggregation operator with neutral value ν we have

$$a_\nu(x_1,...,x_n)=f\left(\sum_{i=1}^{n} f^{\,-1}(x_i)+(1-n)\,f^{\,-1}(\nu)\right) \tag{39}$$

Taking into account the form of the importance transformation

$$x=f\left(wf^{\,-1}(x)\right) \tag{40}$$

and performing the substitution

$$\begin{aligned}a(x_1,...,x_n) &= f\left(\sum_{i=1}^{n} f^{\,-1}\left(f\left(w_i f^{\,-1}(x_i)\right)\right)\right) \\ &= f\left(\sum_{i=1}^{n} w_i f^{\,-1}(x_i)\right)\end{aligned} \tag{41}$$

or taking into consideration the facts that

$$f_\nu(x) = f\left(x + f^{-1}(\nu)\right) \tag{42}$$

$$f_\nu^{-1}(x) = f^{-1}(x) - f^{-1}(\nu) \tag{43}$$

we find that

$$x = f_\nu\left(w f_\nu^{-1}(x)\right) = f\left(w\left(f^{-1}(x) - f^{-1}(\nu)\right)\right) + f^{-1}(\nu)$$

$$\begin{aligned} a_\nu(x_1, ..., x_n) &= f_\nu\left(\sum_{i=1}^{n} f^{-1}(x_i)\right) \\ &= f\left(\sum_{i=1}^{n} w_i\left(f^{-1}(x_i) - f^{-1}(\nu)\right) + f^{-1}(\nu)\right) \\ &= f\left(\sum_{i=1}^{n} w_i f^{-1}(x_i) + \left(1 - \sum_{i=1}^{n} w_i\right) f^{-1}(\nu)\right) \end{aligned} \tag{44}$$

From this form of the weighted aggregation formula it can be seen that if $w_i = 1\,(i = 1, ..., n)$, we regain the starting form and for the w_i values an appropriate restriction is if

$$\sum_{i=1}^{n} w_i = n \tag{45}$$

On the other hand, if we employ the assumption customary in practice, that

$$\sum_{i=1}^{n} w_i = 1 \tag{46}$$

then the aggregation formula is independent of the neutral value ν. Accordingly, an arbitrary expectation level may be chosen.

Below we shall denote the aggregation of the properties $x_1, ..., x_n$ with weights $w_1, ..., w_n$ by

$$a\left(w_1, x_1; w_2, x_2; ...; w_n, x_n\right)$$

Let us examine some special cases of the general construction. If the generator function of the operator is $f(x) = e^{-x}$, then for the conjunction (probability) operator

$$c\left(w_1, x_1; ...; w_n, x_n\right) = \prod_{i=1}^{n} x_i^{w_i} \tag{47}$$

The importance transformation may be illustrated on the following example. Let A and B be properties. If these are equally important, they are mentioned with equal weights. If only property A is important, then with the assumption $\sum_{i=1}^{n} w_i = n$ this means the double inclusion, mention and aggregation of A, and the suppression of property B. This is in accordance with the everyday experience that if some property is considered to be of greater importance, then it is mentioned more often in our argument, while unessential ones are neglected.

This weighting procedure has already been frequently in the course of multicriteria decisions.

A disjunction operator can be obtained with the aid of the operator function $f(x) = 1 - e^{-x}$:

$$d(w_1, n_1; w_2, n_2; ...; w_n, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)^{w_i} \tag{48}$$

This formula is to be found in a paper by Zysno dealing with an empirically based modeling study.

Neglecting that property of the operator that it transforms to the interval $[0, 1]$ as generator function we may choose $f(x) = x$. For the aggregation we then have

$$a(w_1, x_1; w_2, x_2; ...; w_n, x_n) = \sum_{i=1}^{n} w_i x_i \tag{49}$$

This correlation is likewise known in the literature. As a generalization, let the generator function be $f(x) = x^n$, when

$$a(w_1, n_1; ...; w_n, x_n) = \left(\sum_{i=1}^{n} w_i x^{\frac{1}{n}} \right)^n \tag{50}$$

It is worthwhile to mention the correlation that can be formed from the function $f(x) = ke^x / (1 + ke^x)$ generating the rational class of aggregation operators:

$$a(w_1, n_1; ...; w_n, x_n) = \frac{\nu \prod_{i=1}^{n} [(1-\nu) x_i]^{w_i}}{\nu \prod_{i=1}^{n} [(1-\nu) x_i]^{w_i} + (1-\nu) \prod_{i=1}^{n} (\nu (1 - x_i))^{w_i}}$$

In the theory of fuzzy sets the max and min operators are of use. These operators are known to be obtained as the limiting values of the products of conjunction and disjunction operators. The importance transformation is determined on the basis of this limiting value:

$$\begin{aligned} x &= \lim_{\lambda \longrightarrow \infty} f_\lambda \left(w f_{\ \lambda}^{-1}(x) \right) = \lim_{\lambda \longrightarrow \infty} f \left(\left(\left(w f^{\ -1}(x) \right)^\lambda \right)^{1/\lambda} \right) \\ &= \lim_{\lambda \longrightarrow \infty} f \left(w^{1/\lambda} f^{\ -1}(x) \right) = x \end{aligned}$$

Thus, the importance has no effect in the sense of the expansion.

$$\min(w_1, x_1; w_2, x_2; ...; w_n, x_n) = \min(x_1, ..., x_n) \tag{51}$$

Hence, for the min and max operators it may be beneficial to use the importance transformations discussed in the first part of the paper, i.e. $w \longrightarrow x, \min(w, x)$.

5 Determination of the weights

The theory of multi-factorial decisions devotes considerable attention to the methods of determining the data. Naturally, in a concrete decision situation this means very strongly defined possibilities. As essential role is played by the subjective probabilities, including the results of psychology. The methods examine separately the value and the applicable operators. Such a procedure has the advantage that by means of the analysis many incosistencies in human decisions based on numerous parameters and assumptions are eliminated in proportion to the possibilities. However, it has the disadvantages that

(a) performance of the analysis is work-demanding and thus there is no regard to the economy of "time"

(b) small errors made in the independent analysis of the various phases of the decision may become significant in the synthesis

(c) there may be subjective elements in the (individual) phases of the analysis

The use of the method of analysis has the aim of the attainment of objectivity. In the majority of decision procedures, it generally holds that human decisions are more accurate and better than the algorithms of the models. Primarily shape recognition and medical diagnosis may be mentioned, but the position is similar in a decision relating to a traffic situation. In the latter case the economy of time is extremely important. Such types of decisions have the aim of success, and the purpose of the modeling of the decisions is not necessarily the selection of the best alternative, but the achievement of automation.

In such cases there is a possibility for the attainment of decision control and for learning.

Let us consider now a set of objects. (These objects may be fictitious, in the sense that they do not feature among the actual alternatives of the decision, and possess imaginary properties.) Let us denote the evaluation of the j-th property of the i-th object by x_{ij}, and the evaluation of the i-th object by A_j.

Let us assume that these values are known.

Utilizing the form of the aggregation operator we have

$$A_i = f\left(\sum_{j=1}^{n} w_j f^{-1}(x_{ji})\right) \tag{52}$$

Utilizing the strict monotonousness of $f(x)$:

$$f^{-1}(A_i) = \left(\sum_{j=1}^{n} w_j f^{-1}(x_{ji}) \right) \tag{53}$$

If the evaluations are known for $i = 1, ..., n$ and if $\det\left(f^{-1}(x_{ji})\right) \neq 0$, then since the above equation system is linear with respect to the w_j values, the equation system can be solved.

If the number of equations is higher than n, then as the solution method we use the method of least squares to seek the solution for the w_j values.

The method also gives a possibility of using the learning algorithm to determine the values of x_{ji} if these are not known. On the basis of hypothetically assumed values, w_j values are determined, and the value x_{ij} is determined by the importance transformation will then be a better approximation to the actual value than x_{ij}. However, the weighting and evaluation will then be inseparable. Nevertheless, the method can be employed well if the decision relates merely to one class, i.e. the question is only one of whether the alternative belongs in this class or not. With the aid of weights determined in this manner, decisions may be made on the "true" set of alternatives.

6 The problem of weighting

In the theory I have separated the evaluation and the importance of the properties. The whole amounts to the combined consideration of the evaluation and its importance. We have modified the general construction of the operators, in two ways:

1. with the aid of an importance relation, and
2. on the basis of a repetition principle. This latter means that if some property is more important than the others it may be accentuated by repetition of the argument, while those that are insignificant are not mentioned. In the second case the importance weights may also be determined implicitly, by solution of a linear equation system. If the sum of the weights is 1, then the evaluation is independent of the neutral value. To mention a few examples:

If $f(x) = e^{-x}$, then the aggregation is Πx_i, the weighting is $\Pi x_i^{w_i}$, and the scale invariance is $x_i = c_i x_i$.

If $f(x) = x$, then the aggregation is Σx_i, the weighting is $\Sigma w_i x_i$, and the scale invariance is $x_i = x_i + c_i$.

If $f(x) = 1 - e^{-x}$, then the aggregation is $1 - \Pi(1 - x_i)$, the weighting is $1 - \Pi(1 - x_i)^{w_i}$,and the scale invariance is $x_i = (1 - c_i) + x_i c_i$.

The first two examples are in accordance with the utility theory.

To return to the initial critical comments:

1. The membership function is an evaluation, which must be given with consideration to the neutral value. The invariance and the aggregation are closely interdependent.

2. We have not fixed the operators; they satisfy a weak axiom system

3. The results of the decision procedure is not a sharp set. The decision is inter-related with the axiom system. Not only the decision is defined, but the measure of sharpness of the decision is given too.

4. The operators compromise a homogeneous system, including the set implication.

Finally, I would like to mention two interesting features:

1. The aggregative operator is not a logical operator, that is, it can not be obtained as an expression of logical operator systems can be derived from it.

2. The sharpness of the operators of the fuzzy set theory are the fuzziest operators.

7 Summary

In this paper we have defined a general class of fuzzy connectives from which the operator of H. Hamacher and that of R. R. Yager as well as R-fuzzy algebra are obtained as special cases. As an example we have given the construction of a further operator.

We have shown that for the conjunction or disjunction operator a series of operators can be constructed, the limit of which is the min or max operator introduced by L. A. Zadeh.

On the basis of the general construction we have given the connection between the conjunctive, disjunctive and negation operators, which is the necessary and sufficient condition for the fulfillment of DeMorgan identity Thus, by the help of any two operators the third ban be constructed so that the three operators fulfill the DeMorgan identity. From the construction we can obtain H. Hamacher's conditions belonging to the DeMorgan class and R.R. Yager's operator system.

Accordingly, the operators used till now, can be discussed uniformly on the basis of the general construction.

Finally, the measurement of fuzziness was derived from the general construction. (Two fuzzy measurements used up till now were obtained as special cases.) Appropriately the use of the operators is advisable instead of the arbitrary application of different fuzzy measurements. We would like to note here that when the fuzzy theory is applied in decision theory, the optimum is only defined and its degree, i.e. its certainty is not measured. It would be useful to construct a system with the use of which one could conclude the sharpness of a decision from the sharpness of the applied sets and operators.

Our axiomatic system is used by most authors, but no system is in accordance with the results of practical human psychology, because the connection between the membership function and the operators, the problem of importance is not satisfactorily made clear.

Furthermore, the logical operators do not seem to be suitable in human activities.

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[1, 2, 3, 4, 5, 6, 7, 8]

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