A General Class of Fuzzy Operators, the DeMorgan Class of Fuzzy Operators and Fuzziness Measures Induced by Fuzzy Operators

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Abstract

In this paper we examine operators which can be derived from the general solution of functional equations on associativity. We define the characteristics of those functions $f(x)$ which are necessary for the production of operators. We shall show, that with the help of the negation operator for every such function $f(x)$ a function $g(x)$ can be given, from which a disjunctive operator can be derived, and for the three operators the DeMorgan identity is fulfilled. For the fulfillment of the DeMorgan identity the necessary and sufficient conditions are given.

We shall also show that an $f_\lambda(x)$ can be constructed for every $f(x)$, so that for the derived $k_\lambda(x,y)$ and $d_\lambda(x,y)$ $\lim_{\lambda\longrightarrow\infty} k_\lambda(x,y) = \min(x,y)$ and $\lim_{\lambda\longrightarrow\infty} d_\lambda(x,y) = \max(x,y)$.

As Yager's operator is not reducible, for every λ there exist an α, for which, in case $x < \alpha$ and $y < \alpha$, $k_\lambda(x,y) = 0$.

We shall give an $f(x)$ which has the characteristics of Yager's operator, and which is strictly monotone.

Finally we shall show, that with the help of all those $f(x)$, which are necessary when constructing a $k(x,y)$, an $F(x)$ can be constructed which has the properties of the measures of fuzziness introduced by A. De Luca and S. Termini. Some classical fuzziness measures are obtained as special cases of our system.

Keywords: fuzzy operator, DeMorgan identity, fuzziness measure, axiom system

1 Introduction

In fuzzy set theory the basic characteristics of the conjunctive and disjunctive operators, interpreted on fuzzy subsets, are:

1. the operator is associative,
2. the operator is commutative,

3. the correspondence principle is fulfilled, that is, if the fuzzy subset is crisp, then we get the conjunction and disjunction of the classical set theory.

A great number of operators satisfy these criteria, so to make these operators as unambiguously defined as possible, further characteristics of the operators and their Boolean algebraic identity in the fuzzy theory [3], [5], [2], [6] are supposed.

Another group of operators is heuristically constructed and their characteristics are either proved or they correspond with the experimental results, so their usage is advised to the user of fuzzy theory [7], [8].

2 The general class of fuzzy operators

Definition 1 *We say $h(x, y)$ is reducible on both sides, when*

$$h(t, u) = h(t, v) \quad or \quad h(u, w) = h(v, w) \quad only\ if \quad v = u \tag{1}$$

In case of a continuous $h(x, y)$ the reducibility of $h(x, y)$ is equivalent to the strict monotonicity of $\phi_t(u)$ and $\psi_w(u)$, where $\phi_t(u) = h(t, u)$ and $\psi_w(u) = h(u, w)$.

Let us suppose that $u > v \geq 0$, $t \neq 0$. Now in the case of the conjunctive operator $k(t, u) > k(t, v)$, and in the case of the disjunctive operator if $u < v \leq 1$, $w \neq 1$, then $d(u, w) < d(v, w)$.

If we use operators to model decisions and we want to decide on the basis of the two characteristics of the two objectives when the qualities of the two characteristics are equal (t and w), then the operator should be strictly monotone increasing in evaluating the other characteristic, e.g. it chooses the better. It seems to be natural supposition.

The conjunctive operator is reducible on the $(0, 1]$ interval, and the disjunctive operator on the $[0, 1)$ interval, so we are looking for the general solution on the corresponding interval.

Further on we shall be looking for the general form of $k(x, y)$ and $d(x, y)$ supposing they are

1. associative
2. continuous
3. strict monotone on the $(0, 1)$ interval and monotone on the $[0, 1]$ interval.

This last supposition is present in the paper of P.Albert as well as in that of H. Hamacher.

Theorem 2 *(Aczel). If with u and v $h(u, v)$ also always lies in a given (possibly infinite) interval and $h(u, v)$ is reducible on both sides, then*

$$h(x, y) = f\left(f^{-1}(x) + f^{-1}(y)\right) \tag{2}$$

(with continuous and strictly monotone f) is the general continuous solution of

$$h(h(x,y),z) = h(x,h(y,z)). \tag{3}$$

Equation (3) can have reducible continuous solutions only in intervals which are open on one side but which may be open or closed on the other side. From now on we call $f(x)$ the generator function of the operator.

Proof. [1] ■

We shall prove theorems only for the conjunctive operator, but they can be proved for the disjunctive operator on the same way.

Further on we shall examine operators with the $[0,1]$ interval as both the domain and range of the function. Let $f(x)$ be the function assigned to the conjunctive operator $k(x,y)$, and $g(x)$ the one assigned to the disjunctive one $d(x,y)$. We shall define the domain, range and properties of the $f(x)$ and $g(x)$ functions with the help of the following theorems.

We shall determine that function class in the solution of (2) which satisfies the correspondence principle.

As the conjunctive (disjunctive) operator satisfies

$$k(1,0) = k(0,0) = 0 \quad (d(1,0) = d(1,1) = 1), \tag{4}$$

the conjunctive (disjunctive) operator is reducible on the $(0,]\,([0,1))$ interval. According to Aczel's theorem the range of $f(x)\,(g(x))$ and the domain of $f^{-1}(x)\,\left(g^{-1}(x)\right)$ is an $(0,]\,([0,1))$ interval. Because of the continuity of $f(x)\,(g(x))$ the domain and range is an interval open on one side.

$f(x)\,(g(x))$ can be monotonously increasing $(f_i(x), g_i(x))$ as well as monotonously decreasing $(f_d(x), g_d(x))$.

Theorem 3 *Let $k_d(x,y)$ be a conjunctive operator which generator $f_d(x)$ monotonously decreasing function, then there always exist a monotonously increasing generator function $f_i(x)$ for $k_i(x,y)$, for which*

$$k_i(x,y) = k_d(x,y)$$

Proof. If $f_d : [a,b) \longrightarrow (0,1]$ then let $f_i : (-b,-a] \longrightarrow (0,1]$ and $f_i(x) = f_d(-x)$. In this case because of the existence of the inverse $f_i^{-1}(x) = f_d^{-1}(x)$,

$$\begin{aligned} k_d(x,y) &= f_d\left(f_d^{-1}(x) + f_d^{-1}(y)\right) = f_d\left(-f_i^{-1}(x) - f_i^{-1}(y)\right) = \\ &= f_i\left(f_i^{-1}(x) + f_i^{-1}(x)\right) = k_i(x,y) \end{aligned}$$

■

Further we shall consider strictly decreasing $f(x)$ in the case of conjunction and strictly increasing $g(x)$ in the case if disjunction.

Theorem 4 *A monotonously decreasing (monotonously increasing) function $f(x)$ assigned to the conjunctive (disjunctive) operator fulfills*

$$f : [0,\infty) \longrightarrow (0,1] \quad (g : [0,\infty) \longrightarrow [0,1)) \tag{5}$$

Proof. 1. We admit that

$$\lim_{x\longrightarrow 0} f^{\ -1}(x) = \infty \tag{6}$$

Let us suppose that

$$\lim_{x\longrightarrow 0} f^{\ -1}(x) = a, \quad a < \infty$$

which is equivalent to

$$\lim_{x\longrightarrow a} f(x) = 0,$$

because if $x = 0$,

$$0 = k(1,0) \geq k(x,0) \geq k(0,0) = 0,$$

for every y

$$\lim_{x\longrightarrow 0} f\left(f^{\ -1}(x) + f^{\ -1}(y)\right) = 0,$$

from the strict monotonicity and continuity of $f(x)$ this can be fulfilled only if

$$\lim_{x\longrightarrow 0} \left(f^{\ -1}(x) + f^{\ -1}(y)\right) = a.$$

So for every y

$$f^{\ -1}(y) = 0,$$

and this contradicts the strict monotony of y.

2. We admit that

$$f(0) = 1. \tag{7}$$

Let us suppose that $a > 0$, $f(a) = 1$. According to the proof written before $a < \infty$.

So

$$f : [a, \infty) \longrightarrow (0, 1].$$

As

$$1 = k(1,1) = f\left(f^{\ -1}(1) + f^{\ -1}(1)\right) = f(2a) > f(a) = 1,$$

this is a contradiction.

As the conjunctive operator always has a form

$$k(x,y) = f\left(f^{\ -1}(x) + f^{\ -1}(y)\right),$$

its domain is $(0,1] \times (0,1]$, and as $\lim_{x\longrightarrow 0} f^{\ -1}(x) = \infty$, $f^{\ -1}(1) = 0$ so $f^{\ -1} : (0,1] \longrightarrow (\infty, 0]$; this means $f : R^{+} \longrightarrow (0,1]$. ■

So the functions giving conjunctive (disjunctive) operators have the following characteristics (see Fig. 1.)

1.	$f(x)$ is continuous,	$g(x)$ is continuous,
2.	$f(x)$ is strictly mon. decreasing,	$g(x)$ is strictly mon. increasing,
3.	$f : R^+ \longrightarrow (0,1]$,	$g : R^+ \longrightarrow [0,1)$,
4.	$\lim_{x\longrightarrow\infty} f(x) = 0$,	$\lim_{x\longrightarrow\infty} g(x) = 1$,
5.	$f(0) = 1$,	$g(0) = 0$.

(8)

Theorem 5 *Operators made from functions of the properties (8) have the following characteristics:*

1. *commutative:*
$$k(x,y) = k(y,x).$$
2. *associative:*
$$k(k(x,y),z) = k(x,k(y,z)) \quad d(d(x,y),z) = d(x,d(y,z)).$$
3. *monotone: if $y \le z$,*
$$k(x,y) \le k(x,z), \quad d(x,y) \le d(x,z).$$
4. *the correspondence principle is fulfilled:*
$$\begin{aligned} k(1,x) &= x, \quad k(0,x) = 0, \\ d(0,x) &= x \quad d(1,x) = 1 \end{aligned}$$
5. *continuous on the interval $[0,1]$.*
6. $0 \le k(x,y) \le \min(x,y), \quad 1 \ge d(x,y) \ge \max(x,y)$

Proof.

1. $k(x,y) = f\left(f^{-1}(x) + f^{-1}(y)\right) = f\left(f^{-1}(y) + f^{-1}(x)\right) = k(y,x).$
2. $k(k(x,y),z) = f\left(f^{-1}(x) + f^{-1}(y) + f^{-1}(z)\right) = k(x,k(y,z)).$
3. If $y \le z$, then as $f^{-1}(x)$ is monotonously decreasing
$$k(x,y) = f\left(f^{-1}(x) + f^{-1}(y)\right) \le f\left(f^{-1}(x) + f^{-1}(z)\right) = k(x,z).$$
4. $k(x,1) = f\left(f^{-1}(x) + f^{-1}(1)\right) = f\left(f^{-1}(x)\right) = x$
$$k(0,x) = \lim_{y\longrightarrow 0} k(y,x) = \lim_{y\longrightarrow 0} f\left(f^{-1}(x) + f^{-1}(y)\right) = \lim_{z\longrightarrow\infty} f(z) = 0.$$
5. It is fulfilled for $x \in (0,1)$ because $f(x)$ and $f^{-1}(x)$ are continuous, for $x = 0$ and $x = 1$ it comes from (4).
6. Let $\min(x,y) = x$, then
$$k(x,y) \le k(x,1) = x$$

■

3 DeMorgan class of fuzzy operators

As far as the negation is concerned usually the following characteristics are required:

$$n(x) \text{ is continuous} \tag{9}$$

$$n(x) \text{ is strictly monotone decreasing} \tag{9b}$$

$$n(1) = 0 \quad n(0) = 1 \tag{9c}$$

Further we distinguish two forms of the DeMorgan identity:
(1) Conjunctive form:

$$n(k(x, y)) = d(n(x), n(y)) \tag{10}$$

(2) Disjunctive form:

$$n(d(x, y)) = k(n(x), n(y)) \tag{11}$$

Theorem 6 *A disjunctive (conjunctive) operator can be assigned to every conjunctive (disjunctive) operator with the help of the* $n(x)$ *negation.*

Proof. Let $f(x)$ $(g(x))$ be the function in the conjunctive (disjunctive) operator, then we can construct a disjunctive (conjunctive) operator with the help of

$$g(x) = n(f(ax)) \quad (f(x) = n(g(ax))) \quad a \neq 0 \tag{12}$$

because the so defined $g(x)$ $(f(x))$ fulfills the characteristics in (8). ■

Theorem 7 *Every disjunctive (conjunctive) operator, assigned to the conjunctive (disjunctive) operator as in (12), fulfills the conjunctive (disjunctive) form of the DeMorgan identity.*

Proof. If (12) is fulfilled, then

$$n\left(f\left(f^{-1}(x)\right)\right) = g\left(g^{-1}(n(x))\right) = n\left(f\left(ag^{-1}(n(x))\right)\right).$$

As $f(x)$ and $n(x)$ are strictly monotonous,

$$f^{-1}(x) = ag^{-1}(n(x)).$$

This means

$$f^{-1}(x) + f^{-1}(y) = ag^{-1}(n(x)) + ag^{-1}(n(y))$$

as $g(x)$ is strictly monotone,

$$g\left(\frac{f^{-1}(x)+f^{-1}(y)}{a}\right)=g\left(g^{-1}(n(x))+g^{-1}(n(y))\right).$$

As $g(x)=n(f(ax))$ and $a\neq 0$,

$$g\left(\frac{x}{a}\right)=n(f(n(f(x)))),$$

$$n\left(f\left(f^{-1}(x)+f^{-1}(y)\right)\right)=g\left(g^{-1}(n(x))+g^{-1}(n(y))\right),$$

which is the DeMorgan identity. ■

Theorem 8 *Let $k(x,y)$, $d(x,y)$ be conjunctive and disjunctive operators. In this case there can always be given a negation operator $n(x)$, so that the three operators fulfill the conjunctive (disjunctive) DeMorgan identity.*

Proof. Let

$$n(x)=g\left(\frac{f^{-1}(x)}{a}\right)\quad\left(n(x)=f\left(\frac{g^{-1}(x)}{a}\right)\right)\quad a\neq 0 \tag{13}$$

1. $n(x)$ is a strictly monotone decreasing operator if $x_1\leq x_2$: ()

$$g\left(\frac{f^{-1}(x_1)}{a}\right)\geq g\left(\frac{f^{-1}(x_2)}{a}\right)$$

2. $n(x)$ is continuous because $g(x)$ and $f(x)$ are also continuous.

3. $n(1)=0$ and $n(0)=1$ are fulfilled.

   So $n(x)$ has all the characteristics of negation. We shall show, that it fulfills the conjunctive DeMorgan identity.

   From (13)

$$n(f(x))=g\left(\frac{x}{a}\right)\text{ and }\frac{f^{-1}(x)}{a}=g^{-1}(n(x)),$$

   so

$$\begin{aligned}n\left(f\left(f^{-1}(x)+f^{-1}(y)\right)\right)&=g\left(\frac{f^{-1}(x)}{a}+\frac{f^{-1}(y)}{a}\right)\\&=g\left(g^{-1}(n(x))+g^{-1}(n(y))\right)\end{aligned}$$

■

So with the help of any two of the three operators the third can be constructed, and the three operators fulfill the conjunctive (disjunctive) DeMorgan identity. H.Hamacher's DeMorgan class is obtained as a special case. Considering the transformations of the parameters $\gamma, \gamma', \gamma''$ of the operators $k(x,y)$, $d(x,y)$ we can get the relations of $\gamma, \gamma', \gamma''$ for the existence of the DeMorgan class. (In the case $n(x) = 1-x$ the relations, which are valid, are given in Table 1, see section 4.)

Theorem 9 *Let $k(x,y)\,(d(x,y))$ be the conjunctive (disjunctive) operator and $n(x)$ the negation operator. In this case the disjunctive (conjunctive) operator defined by the DeMorgan identity is*
(1) associative
(2) reducible on the interval $[0,1)\,((0,1])$
(3) continuous.

Proof. (1) As the conjunctive operator is associative

$$k(k(x,y),z) = k(x,k(y,z))$$

and so

$$n(k(k(x,y),z)) = n(k(x,k(y,z)))$$

using the conjunctive DeMorgan identity

$$d(n(k(x,y)),n(z)) = d(n(x),n(k(y,z))).$$

Using (10) again

$$d(d(n(x),n(y)),n(z)) = d(n(x),d(n(x),n(z))).$$

Replacing $n(x), n(y), n(z)$ by x, y, z we get that $d(x,y)$ is associative.

(2)From the conjunctive DeMorgan identity

$$d(x,y) = n\left(k\left(n^{-1}(x), n^{-1}(y)\right)\right). \tag{14}$$

Let us suppose that

$$d(x,u) = d(x,v) \tag{15}$$

On the basis of (14)

$$n\left(k\left(n^{-1}(x), n^{-1}(u)\right)\right) = n\left(k\left(n^{-1}(x), n^{-1}(v)\right)\right). \tag{16}$$

As $n(x)$ is strictly monotone, (16) is fulfilled if and only if

$$k\left(n^{-1}(x), n^{-1}(u)\right) = k\left(n^{-1}(x), n^{-1}(v)\right) \tag{17}$$

$k(x,y)$ is reducible on the interval $(0,1]$ so suppose $n^{-1}(x) \neq 0$, then

$$n^{-1}(u) = n^{-1}(v),$$

so $u = v$, and $n^{-1}(x) = 0$ is fulfilled if and only if $x = n(0) = 1$.

As $n(x)$ is strictly monotone, (15) is valid if and only if

$$u = v$$

That is, the disjunctive operator is reducible in the case of $x \neq 1$.

(3)The continuity of $d(x,y)$ is the consequence of the continuity of $k(x,y)$ and $n(x)$. ■

Theorem 10 *Every function which constructs a continuous, associative, reducible operator $h(x,y)$ is unambiguously determined apart from a nonzero constant multiplier of x.*

Proof. [1] ■

Theorem 11 *Let $k(x,y)$, $d(x,y)$ be a conjunctive, disjunctive $n(x)$ a negation operator and let the conjunctive (or disjunctive) DeMorgan identity be fulfilled.*

Then

$$g(x) = n(f(ax)) \qquad (f(x) = n(g(ax))) \tag{18}$$

Proof. Apart from a multiplicative constant the $d(x,y)$ disjunctive operator and its generator $f(x)$ are unambiguously determined on the basis of Theorem 8 and Theorem 9 respectively, and one of the possible solutions is known $(n(f(z)))$,

$$g(cz) = n(f(z)), \quad c \neq 0$$

With the substitution $x = cz$ we get (18). ■

Theorem 12 *If for the negation operator*

$$n(n(x)) = x, \tag{19}$$

then from the conjunctive (disjunctive) DeMorgan identity the validity of the disjunctive (conjunctive) DeMorgan identity follows.

Proof. As

$$n(k(x,y)) = d(n(x), n(y)),$$

and with the substitutions $x := n(x)$, $y := n(y)$ we get

$$n(k(n(x), n(y))) = d(x,y),$$

which is equivalent to

$$k(n(x), n(y)) = n^{-1}(d(x,y)).$$

From (19)

$$n(x) = n^{-1}(x)$$

and from this we get the disjunctive DeMorgan identity.
If

$$n(x) = \left(1 - x^{\lambda}\right)^{1/\lambda},$$

then $n(n(x)) = x$ is fulfilled. ■

4 The operator of L. A. Zadeh is a limes of a succession of operators

Theorem 13 *(1) If $f(x)\,(g(x))$ satisfies conditions (8), then*

$$f_{\lambda}(x) = f\left(x^{1/\lambda}\right) \quad \left(g_{\lambda}(x) = g\left(x^{1/\lambda}\right)\right) \tag{20}$$

satisfies them too.

(2) If the inverse of $f(x)\,(g(x))$ is $f^{-1}(x)\left(g^{-1}(x)\right)$ then the inverse of $f_{\lambda}(x)\,(g_{\lambda}(x))$ is

$$f_{\lambda}^{-1}(x) = \left(f^{-1}(x)\right)^{\lambda} \quad \left(g_{\lambda}^{-1}(x) = \left(g^{-1}(x)\right)^{\lambda}\right).$$

Proof. Trivial. ■

Let

$$k_{\lambda}(x,y) = f_{\lambda}\left(f_{\lambda}^{-1}(x) + f_{\lambda}^{-1}(y)\right) = f\left(\left(\left(f^{-1}(x)\right)^{\lambda} + \left(f^{-1}(y)\right)^{\lambda}\right)^{1/\lambda}\right),$$

$$d_{\lambda}(x,y) = g_{\lambda}\left(g_{\lambda}^{-1}(x) + g_{\lambda}^{-1}(y)\right) = g\left(\left(\left(g^{-1}(x)\right)^{\lambda} + \left(g^{-1}(y)\right)^{\lambda}\right)^{1/\lambda}\right)$$

Theorem 14

$$\lim_{\lambda \longrightarrow \infty} k_{\lambda}(x,y) = \min(x,y) \quad \left(\lim_{\lambda \longrightarrow \infty} d_{\lambda}(x,y) = \max(x,y)\right) \tag{21}$$

Proof. Let us suppose that $x \leq y$. As $f(x)$ is monotonously decreasing

$$c = \frac{f^{-1}(y)}{f^{-1}(x)} \leq 1.$$

So

$$
\begin{aligned}
\lim_{\lambda \longrightarrow \infty} k_\lambda (x,y) &= \lim_{\lambda \longrightarrow \infty} f\left(f^{\ -1}(x)\left(1+\left(\frac{f^{\ -1}(y)}{f^{\ -1}(x)}\right)^\lambda\right)^{1/\lambda}\right) \\
&= f\left(f^{\ -1}(x)\lim\left(1+c^\lambda\right)^{1/\lambda}\right) \\
&= f\left(f^{\ -1}(x)\exp\left(\lim_{\lambda \longrightarrow \infty}\frac{1}{\lambda}\ln\left(1+c^\lambda\right)\right)\right) \\
&= f\left(f^{\ -1}(x)\exp\left(\lim_{\lambda \longrightarrow \infty}\frac{k}{\lambda}\right)\right) \\
&= f\left(f^{\ -1}(x)\exp(0)\right) = x = \min(x,y)
\end{aligned}
$$

where we made use of $\ln\left(1+c^\lambda\right) \le k$ if $c \le 1$. ■

5 Special operators and their characteristics

Consequences:

1. Yager's operator can be derived from the operator

$$f(x) = 1 - x$$

2. The operator we proposed can be derived from the operator

$$f(x) = \frac{1}{1+x}$$

Theorem 15 *If*

$$x, y \le \alpha = 1 - \left(\frac{1}{2}\right)^{1/\lambda}$$

then Yager's intersection operator is

$$k_\lambda^\gamma (x,y) = 0$$

Proof. As it can be easily seen

$$
\begin{aligned}
\left((1-x)^\lambda + (1-y)^\lambda\right)^{1/\lambda} &\ge \left(\left(1-\left(1-\left(\frac{1}{2}\right)^{1/\lambda}\right)\right)^\lambda + \left(1-\left(1-\left(\frac{1}{2}\right)^{1/\lambda}\right)\right)^\lambda\right)^{1/\lambda} \\
&= 1.
\end{aligned}
$$

Yager's operator is of the following form

$$k_{\lambda}_{x,y\leq\alpha}(x,y) = 1 - \min_{x,y\leq\alpha}\left(1, \left((1-x)^{\lambda} + (1-y)^{\lambda}\right)^{1/\lambda}\right) == 1 - 1 = 0$$

■

Table 1 lists the characteristics of the operators proposed by R. R. Yager, H. Hamacher and ourselves (our negation operation is $n(x) = 1 - x$)

We note here, that

1. Yager's operator does not have all the characteristics of (9). In spite of the theorems of this paper are valid as the proofs show.

2. We can obtain the form of the operator in H. Hamacher's paper, if we substitute parameter λ occurring in our table for $1/\gamma$ in the case of conjunctive operator, and for $1/(\gamma' + 1)$ in the case of disjunctive operator. In the resulting transformed form the condition of the fulfillment of the DeMorgan identity is $1/\gamma = 1/(\gamma' + 1)$, which is equivalent to the results of H. Hamacher in case $n(x) = 1 - x$.

3. It can be shown that Yager's formula is equivalent to

   (a)
   $$1 - \min\left(1, \left((1-x)^{\lambda} + (1-y)^{\lambda}\right)^{1/\lambda}\right)$$

   (b)
   $$\min\left(1, \left(x^{\lambda} + y^{\lambda}\right)^{1/\lambda}\right)$$

4. In the case of all three operators if $\lambda \leq \lambda'$, then

$$k_{\lambda}(x,y) \leq k_{\lambda}(x,y) \quad (d_{\lambda}(x,y) \geq d_{\lambda}(x,y))$$

5. In the case of all three operators

$$\lim_{\lambda\longrightarrow 0} k_{\lambda}(x,y) = 0 \quad \left(\lim_{\lambda\longrightarrow 0} d_{\lambda}(x,y) = 1\right)$$

6 Fuzziness measure induced by the fuzzy operator

In 1972 A. DeLuca and S. Termini [4] introduced a measure $d(\mu)$, reflecting the fuzziness of the membership function $\mu(t_i), i = 1, 2, ..., N$. Their requirements of a such a measure are:

$P_1 : d(\mu)$ must be 0 if and only if $\mu(t_i)$ takes on values 0 or 1.

$P_2 : d(\mu)$ must assume the maximum value if and only if $\mu(t_i)$ always takes the value $\frac{1}{2}$.

P_3 : $d(\mu)$ must be greater than or equal to $d(\mu^*)$, where $\mu^*(t_i)$ is any sharpened version of $\mu(t_i)$, that is

$$\mu^*(t_i) \geq \mu(t_i) \text{ if } \mu(t_i) \geq \frac{1}{2}$$

$$\mu^*(t_i) \leq \mu(t_i) \text{ if } \mu(t_i) \leq \frac{1}{2}$$

In this section we will construct a general fuzziness measure which is induced by the conjunctive operators and of which the generator $f(x)$ satisfies the properties (8) and is strictly convex and derivable.

Let

$$d(\mu) = \frac{1}{N}\sum_{i=1}^{N} F(\mu(t_i))$$

where

$$F(x) = \frac{1}{f\left(2f^{-1}\left(\frac{1}{2}\right)\right)} f\left(f^{-1}(x) + f^{-1}(1-x)\right) = \frac{1}{f\left(2f^{-1}\left(\frac{1}{2}\right)\right)} k(x, 1-x).$$

Theorem 16 *The fuzziness measure defined above satisfies the properties* P_1, P_2, P_3.

Proof.

1. $0 \leq F(x) \leq 1$.

   (a) As $f(x)$ is always positive: $0 \leq F(x)$

   (b) As $f^{-1}(x)$ is strictly convex,

   $$\frac{f^{-1}(a) + f^{-1}(b)}{2} \geq f^{-1}\left(\frac{a+b}{2}\right).$$

   Let $a = x$ and $b = 1 - x$, then

   $$f^{-1}(x) + f^{-1}(1-x) \geq 2f^{-1}\left(\frac{1}{2}\right)$$

   So $F(x) \leq 1$.

2. (P_1) If $x = 0$ or $x = 1$, then $F(x) = 0$ comes from identity

   $$k(0,1) = k(1,0) = 0.$$

3. (P_2) $F(x) \leq 1$ and $F\left(\frac{1}{2}\right) = 1$.

4. (P_3) Let us suppose that $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$; then $F(x_1) \leq F(x_2)$. As $f(x)$ is strictly decreasing it is enough to show that

$$f^{-1}(x_1) + f^{-1}(1 - x_1) \geq f^{-1}(x_2) + f^{-1}(1 - x_2)$$

which is equivalent to the fact that the function

$$G(x) = f^{-1}(x) + f^{-1}(1 - x)$$

is strictly decreasing in the interval $\left[0, \frac{1}{2}\right]$. Now we shall prove that $G'(x) \leq 0$, that is

$$\left(f^{-1}(x)\right)' \leq \left(f^{-1}(y)\right)'_{y=1-x} \tag{22}$$

If $0 \leq x \leq \frac{1}{2}$, then $x \leq y$. So (23) is fulfilled in the case of a convex, monotonously decreasing, continuous function $f(x)$. ■

To $f_\lambda(x)$ defined in (20) one can always construct a corresponding $F_\lambda(x)$.

Theorem 17

$$\lim_{\lambda \longrightarrow \infty} F_\lambda(x) = 2 \min(x, 1 - x). \tag{23}$$

Proof. As $\lim_{\lambda \longrightarrow \infty} k_\lambda(x, y) = \min(x, y)$, it is sufficient to see that

$$\begin{aligned} 2 &= \lim_{\lambda \longrightarrow \infty} \frac{1}{f\left(\left(2\left(f^{-1}\left(\frac{1}{2}\right)\right)^\lambda\right)^{1/\lambda}\right)} = \\ &= \frac{1}{f\left(2^{1/\lambda} f^{-1}\left(\frac{1}{2}\right)\right)} = \frac{1}{f\left(f^{-1}\left(\frac{1}{2}\right)\right)} = 2 \end{aligned}$$

■

Theorem 18 *If $f(x) = e^{-x}$, then*

$$F(x) = 4x(1 - x) \tag{24}$$

Proof. Trivial ■

Summary 19 *In this paper we have defined a general class of fuzzy connectives from which the operator of H. Hamacher and that of R. R. Yager as well as R-fuzzy algebra are obtained as special cases. As an example we have given the construction of a further operator.*

We have shown that for the conjunction resp. disjunction operator a series of operators can be constructed, the limit of which is the min resp. max operator introduced by L. A. Zadeh.

On the basis of the general construction we have given the connection between the conjunctive, disjunctive and negation operators, which is the necessary and

sufficient condition for the fulfillment of the DeMorgan identity. So by the help of any two operators the third can be constructed so that the three operators fulfill the DeMorgan identity. On the basis of the construction we can obtain Hamacher's conditions belonging to the DeMorgan class and Yager's operator system.

So the operators used till now can be discussed uniformly on the basis of the general construction.

Finally, the measurement of fuzziness was derived from the general construction. (Two fuzzy measurements, used up till now, were obtained as special cases.). So, furthermore, it is advisable to use the operators appropriately instead of the arbitrary application of different fuzzy measurement. We would like to note here that when the fuzzy theory is applied in decision theory, the optimum is only defined and its degree, i.e. its certainty are not measured. It would be useful to construct a system in the use of which one could conclude the sharpness of decision from the sharpness of the applied sets and operators.

Our axiomatic system is used by most authors, but no system is in accordance with the results of practical human physiology, because the connection between membership function and the operators, the problem of importance, is not satisfactorily made clear.

Furthermore, in human activities it is not the logical operators which seem to be suitable.

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Conjunction

Author	$f(x) =$	$f^{-1}(x) =$	$k_\lambda(x,y) =$	$k_1(x,y) =$	$\lim_{\lambda\longrightarrow\infty} k_\lambda(x,y) =$
H. Hamacher	$\frac{e^{-x}}{\lambda+(1-\lambda)e^{-x}}$	$-\ln\frac{\lambda x}{1+(\lambda-1)x}$	$\frac{\lambda xy}{1-(1-\lambda)(x+y-xy)}$	xy	$\frac{xy}{x+y-xy}$
R. R. Jager	$1-x^{1/\lambda}$ if $x<1$,	$(1-x)^\lambda$ if $x<1$	$1-\left((1-x)^\lambda+(1-y)^\lambda\right)^{1/\lambda}$	$\max(0,x+y-1)$	$\min(x,y)$
	0 if $x\geq 1$	0 if $x\geq 1$	if $(1-x)^\lambda+(1-y)^\lambda<1$		
			0 if $(1-x)^\lambda+(1-y)^\lambda\geq 1$		
J. Dombi	$\frac{1}{1+x^{1/\lambda}}$	$\left(\frac{1}{x}-1\right)^\lambda$	$\frac{1}{1+\left(\left(\frac{1}{x}-1\right)^\lambda+\left(\frac{1}{y}-1\right)^\lambda\right)^{1/\lambda}}$	$\frac{xy}{x+y-xy}$, 0 if $x=0, y=0$	$\min(x,y)$

Disjunction

	$g(x) =$	$g^{-1}(x) =$	$d_\lambda(x,y) =$	$d_1(x,y) =$	$\lim_{\lambda\longrightarrow\infty} d_\lambda(x,y) =$
H. Hamacher	$\frac{\lambda\left(1-e^{-x}\right)}{\lambda+(1-\lambda)e^{-x}}$	$-\ln\frac{\lambda(1-x)}{\lambda-(\lambda-1)x}$	$\frac{\lambda(x+y)+xy(1-2\lambda)}{\lambda=xy(1-\lambda)}$	$x+y-xy$	$\frac{x+y-2xy}{1-xy}$
R. R. Yager	$x^{1/\lambda}$ if $x<1$	x^λ if $x<1$	$\left(x^\lambda+y^\lambda\right)^{1/\lambda}$ if $x^\lambda+y^\lambda<1$	$\min(1,x+y)$	$\max(x,y)$
	1 if $x\geq 1$	1 if $x\geq 1$	1 if $x^\lambda+y^\lambda\geq 1$		
J. Dombi	$\frac{1}{1+x^{-1/\lambda}}$	$\left(\frac{1}{x}-1\right)^{-\lambda}$	$\frac{1}{1+\left(\left(\frac{1}{x}-1\right)^{-\lambda}+\left(\frac{1}{y}-1\right)^{-\lambda}\right)^{-1/\lambda}}$	$\frac{x+y-2xy}{1-xy}$, 1 if $x=1, y=1$	$\max(x,y)$

References

[1] J. Aczél. *Lectures on Functional Equations and Their Applications*. Academic Press, New York, 1966.

[2] P. Albert. The algebra of fuzzy logic. *Fuzzy Sets and Systems*, 1:203–230, 1978.

[3] R. Bellman and M. Giertz. On the analytic formalism of the theory of fuzzy sets. *Information Science*, 5:149–156, 1973.

[4] A. DeLuca and S. Termini. A definition of a non-probabilistic entropy in the setting of fuzzy sets theory. *Inform and Control*, pages 301–312, 1972.

[5] L. W. Fung and K. S. Fu. An axiomatic approach to rational decision making in a fuzzy environment. In L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, editors, *Fuzzy Sets and their Applications to Cognitive and Decision Processes*. Academic Press, New York, 1975.

[6] H. Hamacher. *Über Logische Agregationen Nicht Binar Explizierter Entscheidungskriterien*. Rita G. Fischer Verlage, Frankfurt am Main, 1978.

[7] R. R. Yager. On a general class of fuzzy connectives. Technical report, Iona College.

[8] H.-J. Zimmermann, U. Thole, and P. Zysno. On the suitability of minimum and product of operators for the interpretation of fuzzy sets. *Fuzzy Sets and Systems*, 2:167–180, 1979.